UNIVERSITAT DE BARCELONA

Regularity theory for obstacle problems and boundary Harnack inequalities

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by

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To my people,
for always being there.

This Thesis is left as an exercise to the reader.[1]

[1] This is a joke.

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[2] They were my roommate.

Summary

This Thesis is dedicated to the study of ellipitic and parabolic Partial Differential Equations, both local and nonlocal. More specifically, this work concerns the regularity properties of some obstacle problems.

The obstacle problem in its simplest form is the following nonlinear elliptic problem:

$$\begin{cases} u & \geq & 0 & \text{in} & \Omega \\ \Delta u & \leq & 1 & \text{in} & \Omega \\ \Delta u & = & 1 & \text{in} & \{u > 0\} \cap \Omega \end{cases}$$

It is a canonical example of a free boundary problem, that is, a PDE problem where the unknowns are not only a function u, but also a subdivision of the domain into different regions, in this case, $\{u > 0\}$ and $\{u = 0\}$, and the PDE satisfied in each region is different.

Free boundary problems are a very active field of research. On the one hand, free boundaries are a good model for interfaces in real-world settings, with applications in Physics, Biology, Finance and Engineering. On the other hand, they have been a source of interesting mathematical challenges, motivating the fine analysis of solutions to ellipitic and parabolic equations.

This Thesis is divided into two Parts. Part I is devoted to the study of several different obstacle problems.

In Chapter 1, we study the obstacle problem for parabolic operators of the type $\partial_t + L$, where L is an elliptic integro-differential operator of order $2s$, such as $(-\Delta)^s$, in the supercritical regime $s \in (0, \frac{1}{2})$. We establish the optimal $C^{1,1}$ regularity of solutions, which is surprisingly *better* than in the elliptic problem, and we also show that the free boundary is *globally* $C^{1,\alpha}$.

Our main difficulties are the lack of monotonicity formulas, and the supercritical scaling of the equation, that is, the fact that the highest order of differentiation corresponds to the time derivative, which renders the usual techniques impossible to use. To overcome them, we use barriers, scaling arguments, and the key observation that $u_t = (Lu)^-$, that allows us to derive time regularity from space regularity.

Chapter 2 is devoted to the generic regularity properties of the free boundary in the thin obstacle problem. Since there are many pathological examples of solutions to free boundary problems, often the goal is instead of proving regularity for all solutions, proving regularity for *most* solutions in an appropriate sense.

In our work, we show that, for one-parameter monotonous families of solutions, for almost every solution, the free boundary is smooth outside of a set of codimension $2 + \alpha_0$ (in the free boundary). In particular, this means that in $\mathbb{R}^3$ and $\mathbb{R}^4$, the free boundary is generically smooth.

We conclude Part I with Chapter 3, where we use a nonlocal analogue of the Bernstein technique to establish semiconvexity estimates for a wide class of nonlinear nonlocal ellipitic and parabolic equations, including obstacle problems. As a consequence, we extend

the known regularity theory for nonlocal obstacle problems in the full space to problems in bounded domains.

In Part II, we extend the boundary Harnack inequality to (local) elliptic and parabolic equations with a right-hand side.

The boundary Harnack is a classical result in potential theory that states that if u and v are positive harmonic functions that vanish on part of the boundary of a regular enough domain, then u/v is bounded and Hölder continuous up to the boundary. Boundary Harnack inequalities are used in the proof of the smoothness of free boundaries in several obstacle problems, in the key step of seeing that if a free boundary is flat Lipschitz, then it is $C^{1,\alpha}$.

The goal of our work was to extend the regularity theory of obstacle problems to the fully nonlinear setting. To do so, we developed boundary Harnack inequalities for equations in non-divergence form with a right-hand side.

Chapter 4 concerns elliptic equations and Chapter 5 is about parabolic equations. The techniques used are different. In the elliptic setting, it is enough to use barriers, scaling arguments and a standard iteration to deduce the Hölder regularity of the quotient. However, in the parabolic world, the proofs are much more involved and they are based on a delicate contradiction-compactness argument.

This Thesis is divided into two Parts. Each Part is divided into Chapters. Each Chapter corresponds to a paper or a preprint, as follows:

Part I:

- X. Ros-Oton, C. Torres-Latorre, *Optimal regularity for supercritical parabolic obstacle problems*, Comm. Pure Appl. Math. **77** (2024), 1724-1765.
- X. Fernández-Real, C. Torres-Latorre, *Generic regularity of free boundaries for the thin obstacle problem*, Adv. Math. **433** (2023), 109233.
- X. Ros-Oton, C. Torres-Latorre, M. Weidner, *Semiconvexity estimates for nonlinear integro-differential equations*, preprint arXiv (2023).

Part II:

- X. Ros-Oton, C. Torres-Latorre, *New boundary Harnack inequalities with right hand side*, J. Differential Equations **288** (2021), 204-249.
- C. Torres-Latorre, *Parabolic boundary Harnack inequalities with right-hand side*, Arch. Rational Mech. Anal. (2024), in press.

Contents

Introduction

This Thesis is devoted to the study of regularity theory for elliptic and parabolic free boundary problems, both local and nonlocal... but let us start from the beginning.

A very old and central question in mathematics is that of solving equations, that is, given an equality with one or several unknowns, determine the value of these unknowns.

When we think of an equation, we usually think of an expression with an equal sign and an unknown x, and we think of the process of solving the equation as finding the values of x that make the equality true. But even with simple examples like $x(x+1) = 1$, we must consider this equation in the context of the real problem it comes from.

For example, if we want to find a rectangle of unit area, whose sides differ by a unit length, then x must be a positive number and our equation from before has only one solution. But if we consider the task of finding a real number x such that $x(x+1) = 1$, then we find two solutions.

It should be clear by now that when one deals with an equation, it is also necessary to understand where are we allowed to look for a solution, and what is the right notion of solution. This is a recurring topic in the theory of partial differential equations.

Partial differential equations

Partial differential equations (in short, PDE) are equations where the unknown u is a function of several variables, and there are expressions involving partial derivatives in the equality. The study of PDE comes originally from the study of evolution equations in physics, such as the Euler Equation, that describes the motion of incompressible fluids, and the Heat Equation, that describes the evolution of temperature when conduction is the dominant behaviour.

For example, if $u(x,t)$ is the temperature of a solid $\Omega \subset \mathbb{R}^3$, then

$$\frac{\partial u}{\partial t} = c\Delta u := c\sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2}.$$

In order to compute u, we need to know the temperature at an initial state $u(x,0)$, an *initial condition*, and also the temperature or the heat flux on the surface of the body, $u(x,t)$ for $x \in \partial\Omega$, a *boundary condition*.

Some examples of PDE

Many PDE appear in physics problems, such as:

- The heat (or diffusion) equation.
$$u_t = \Delta u.$$
- The Poisson equation in electrostatics.
$$\Delta u = f.$$
- The wave equation in ondulatory mechanics.
$$u_{tt} = \Delta u.$$
- The Schrödinger equation in quantum mechanics.
$$u_t = -i\Delta u.$$
- The Maxwell equations of electromagnetism.
- The Euler and Navier-Stokes equations in fluid dynamics.

But PDE also come from other areas of science where functions of several variables are involved, for example:

- The reaction-diffusion equation in epidemiology and population dynamics.
$$u_t = \Delta u + f(u).$$
- The Burgers equation in traffic dynamics.
$$u_t = -uu_x.$$
- The Black-Scholes equation in finance.

Finally, PDE appear in problems from other areas of mathematics:

- The Cauchy-Riemann equations in complex analysis.
$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$
- The minimal surface equation in differential geometry.
$$\mathrm{Div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0.$$
- The nonlocal diffusion equation in stochastic calculus.

Theory of PDE

Much unless ordinary differential equations, PDE do not have a general unified theory. On the contrary, given the vast amount of examples and how differently they behave, it seems unreasonable to hope for a useful unified approach. Despite needing specialized treatment, there are several overarching principles and historical trends that are transversal in the theory of PDE.

Explicit formulas

The first approach to solving an equation is trying to find a formula for the solution. Even if this is impossible to do in full generality, there are examples of domains with high symmetry where the study of PDE can be reduced to the study of one-dimensional problems, and then we can take advantage of the techniques to solve ordinary differential equations.

For example, we can solve the Laplace equation in the unit ball with the Poisson kernel representation. If $\Delta u = 0$ in the unit ball B_1 and $u = g$ on ∂B_1, then

$$u(r,\theta) = \int_{\partial B_1} \frac{1-r^2}{1-2r\cos(\theta-\xi)+r^2} g(\xi) \mathrm{d}\xi$$

in polar coordinates. When g is smooth, it follows from the formula that u is smooth as well, making the expression $\Delta u = 0$ meaningful.

Other examples of PDE that admit explicit solutions are the linear transport, wave, and heat equations, in some domains. Such examples are useful in applications, and they also give insight on what behaviours we can expect for solutions in more general domains.

Separation of variables

A method that is very used in physics and engineering is separation of variables. We will illustrate how this method operates for the problem of heat conduction in a one dimensional rod. Let $u(x,t)$ be the temperature as a function of space and time, for $x \in (0,\pi)$, and assume $u(0) = u(\pi) = 0$ and that the initial temperature is $u(x,0) = u_0(x)$. The evolution of the temperature is given by the heat equation $u_t = \Delta u$.

Now, consider solutions of the form $X(x)T(t)$. Then,

$$X(x)T'(t) = X''(x)T(t),$$

and so

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

The key point here is that the function in the left only depends on t, while the right-hand side only depends on x. Therefore, they are constant. This is the key observation that gives name to the method.

The next step is considering solutions to $X''(x) = \lambda X(x)$ such that $X(0) = X(\pi) = 0$ to take into account the boundary conditions. Here we can use ordinary differential equation methods to find that the only solutions are $X(x) = \sin(nx)$ with $\lambda = -n^2$ and n a positive integer. Then, we use this knowledge in $T'(t) = -n^2 T(t)$ to obtain $T(t) = c_n(x)e^{-n^2 t}$. Combining the information,

$$X(x)T(t) = c_n \sin(nx)e^{-n^2 t}.$$

Finally, since the equation is linear, we can write solutions as linear combinations of these product solutions:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx) e^{-n^2 t}.$$

There are important missing pieces here:

- Giving an adequate meaning to the infinite sum.
- Proving that all solutions have this form.
- Computing the coefficients c_n, using the fact that at $t = 0$,

$$u_0(x) = \sum_{n=1}^{\infty} c_n \sin(nx).$$

Historically, these problems motivated the development of the Fourier series, and giving it a rigorous meaning was a driving force behind many of the advances in mathematical analysis in the 19th century.

Power series

While not very used in practice, power series give a local existence and uniqueness result for analytic PDE that is completely general.

Theorem (Cauchy-Kovalevskaya theorem)**.** *Let $A_1, \dots, A_{n-1}, b : \mathbb{R}^{n+1} \to \mathbb{R}$ be analytic. Then, there exists a neighbourhood V of the origin in $\mathbb{R}^n$ such that the problem*

$$\begin{cases} u_n &= A_1(x,u)u_1 + \ldots + A_{n-1}(x,u)u_{n-1} + b(x,u) & \text{in} \quad V \\ u &= 0 & \text{on} \quad \{x_n = 0\} \cap V \end{cases}$$

has a unique analytic solution $u : V \to \mathbb{R}$.

There are also higher order, vectorial, and complex variable versions of this theorem. The proof is based on expanding the functions appearing in the PDE as formal power series and deriving expressions for the coefficients. Hence, it is only available for analytic functions.

Classical solutions

Short of a formula, the most intuitive notion of a solution to a given PDE is a function u of class C^k, where k is the highest order of differentiation that appears in the PDE, such that the equality holds in a pointwise sense. Moreover, to have a meaningful notion of solution, we will ask it to have three properties:

- Existence.
- Uniqueness.
- Stability, i.e. the solution should change continuously with respect to perturbations in the initial or the boundary data.

When a class of PDE has solutions satisfying these conditions, we say the problem is well-posed.

It turns out that this notion of solution is, in general, too restrictive. In many cases the physical solutions to a problem will be only continuous or Lipschitz, but the PDE will have higher order derivatives, making the existence of classical solutions impossible. Even in the cases where classical solutions do exist, finding them directly is a very hard task, because of the bad compactness properties of C^k.

Generalized solutions

In many applications, finding a classical solution is not only mathematically hard, but is just not possible, or it fails to capture behaviours that we want to model. For example, if we consider the transport equation

$$u_t = u_x$$

with discontinuous initial data, the physically meaningful solution would be $u(x,t) = u(x+t,0)$ as in the smooth case, even if it no longer makes sense to evaluate the equation pointwise.

There are several approaches to this problem, namely, semigroup methods for evolution equations, weak solutions, and viscosity solutions. Each one has advantages and disadvantages, and they are available or not depending on the particular PDE. While these *solutions* do not solve the PDE pointwise, we still ask them to have the three properties that make the problem well-posed.

Descriptive results

Once there is a suitable notion of solution that makes a problem well-posed, the next step is studying the properties of solutions. We list here some broad classes of descriptive results:

- **Regularity**: given a (generalized) solution to a PDE, it belongs to a better function space. This usually comes with an estimate on the norm in such space.
- **Asymptotics**: given a solution to a PDE, it has some rate of decay at infinity, or it converges to a special profile.
- **Rigidity**: given a solution to a PDE in a particular domain, it belongs to a much more restrictive set of functions. For example, all positive harmonic functions in $\mathbb{R}^n$ are constant.

In this Thesis, we will focus on the regularity question.

Elliptic and parabolic PDE

Diffusion is the movement of anything (heat, bacteria, money...) from where there is more quantity to where there is less. In a very broad sense, diffusion is usually a consequence of the second law of thermodynamics (entropy increases over time), and hence we can expect it to be a very common phenomenon in the real world.

Let us build a mathematical model of diffusion. We will consider $u(x,t)$ a function of space and time, representing the concentration of what is diffusing. Then, we have the diffusion equation

$$u_t = Lu,$$

where L is the diffusion operator. Now, we have an important choice: L can be local or nonlocal, meaning that we can consider that diffusion happens at small scales, and what happens far away does not affect immediately, which is more appropriate for modelling heat transfer for example, or we can allow *jumping* and long-range interactions, which would be more appropriate for a model of the stock market[3].

For simplicity, let L be local. Now, the rate of growth of u at a certain point will be an average of the inflow minus the outflow in all directions. Let us call $\mathbf{q} = (q_1, \ldots, q_n)$ the flux of u, which of course is also a function of space and time. Then, we can write the fact that the variation of u corresponds to the quantity that enters minus the quantity that leaves as

$$u_t = \int_{|e|=1} -e \cdot \partial_e \mathbf{q} = -\operatorname{Div}(\mathbf{q}).$$

This is sometimes called the continuity equation.

On the other hand, we want to take into account the fact that the flow goes from the places where u is higher to where u is lower, and that it is larger when the difference is bigger. This can be formulated as Fick's first law of diffusion

$$\mathbf{q} = -\nabla u.$$

Combining the continuity equation with Fick's law of diffusion, we obtain the diffusion (or heat) equation

$$u_t = \Delta u,$$

so the operator L that we were looking for is the Laplace operator. We will call *parabolic equations* to evolution PDE that are driven by a diffusive operator such as the Laplacian.

Now, imagine that we are interested in what will happen after a long time of diffusion, once the system has reached some equilibrium. This does not mean that the flux $\mathbf{q}$ is zero, but rather that at every point the entering and exiting fluxes compensate each other. We can also think of equilibrium as $u_t = 0$, that is, u is not changing anymore over time. In any case, we obtain an equation of the form

$$Lu = 0,$$

where L is a diffusion operator. Such stationary-state equations are called *elliptic equations*.

Notice that in our deduction we are assuming, among other simplifications, that the process is isotropic (does not depend on the direction), translation-invariant (does not depend on the point in space), and linear (diffusion speed only depends on the difference of values of u, and not on the total value of u). While these assumptions are reasonable, dropping them gives rise to a richer class of PDE that still have the fundamental properties of diffusion.

Harmonic functions

Harmonic functions are solutions of the Laplace equation,

$$\Delta u = 0.$$

As steady states of isotropic diffusion, harmonic functions play a central role in mathematical physics and engineering. They also appear in many areas of mathematics, for example, as the

[3]Internet memes are also thought to be subject to nonlocal diffusion, but more research is needed.

real and imaginary parts of holomorphic functions, or as minimizers of the Dirichlet energy functional.

Harmonic functions are characterized by the fact that the value at each point equals the average of its neighbouring values.

Proposition (Mean Value Property)**.** *Let U be an open subset of $\mathbb{R}^n$, and let $u : U \to \mathbb{R}$ be a continuous function. Then, $\Delta u = 0$ in U if and only if for every $x \in U$, and for every $r \in (0, \mathrm{dist}(x, \partial U))$,*

$$u(x) = \fint_{B_r(x)} u.$$

Poisson kernel representation

Given a harmonic function in a bounded reasonable[4] domain, we can recover the values in the interior from the values on the boundary through the Poisson kernel representation. If u is harmonic in Ω, and continuous in $\overline{\Omega}$, then

$$u(x) = \int_{\partial\Omega} u(y) P_x(y) \mathrm{d}y.$$

The Poisson kernel $P_x(y)$ is defined on $\partial\Omega$. It can be computed explicitly in some particular cases like the ball.

Local elliptic operators

Divergence form operators

If we want to include anisotropic effects in our equation, we may put coefficients that depend on the point and the direction. For example, if we want to use Fick's law but now the flux is not just the gradient of u, but rather $\mathbf{q} = A(x)\nabla u$, where $A(x)$ represents how the material is different in different directions, we obtain *divergence form elliptic operators*,

$$Lu := \mathrm{Div}(A(x)\nabla u). \tag{1}$$

Now, we cannot allow for any $A(x)$. For example, if we take minus the identity, the flux would point towards where there is more u, which is not a diffusive behaviour anymore. A reasonable assumption on $A(x)$ is one that would make the flux point *not backwards*, that is,

$$\langle A(x)\nabla u, \nabla u\rangle = \langle \mathbf{q}, \nabla u\rangle \geq 0,$$

and since we want this to happen for any possible value of ∇u, this is the same as asking $A(x)$ to be semidefinite positive. With this assumption on $A(x)$, we call L *degenerate elliptic*. Note that this still allows the flux to be perpendicular to the gradient of u.

In most physical applications, we actually want the flux to point towards where u is smaller, and then the condition is replaced by

$$\langle A(x)\xi, \xi\rangle \geq \lambda|\xi|^2,$$

[4] See for example [163] for details.

together with $A(x)$ being bounded. With these two conditions on $A(x)$, we call L a *uniformly elliptic divergence form operator.*

Divergence form equations have a *variational structure*, that is, they are related to the minimization of energies. Indeed, if we want to find a function u that minimizes the energy

$$E(u) = \int_\Omega \nabla u^\top A(x) \nabla u,$$

subject to $u = g$ on the boundary of Ω, and A is a symmetric matrix, then u is a solution[5] to

$$\mathrm{Div}(A(x)\nabla u) = 0 \quad \text{in } \Omega.$$

Non-divergence form operators

Let us recall the following identity:

$$\Delta u = \mathrm{Div}(\nabla u) = \mathrm{Tr}(D^2 u).$$

We already used the first reformulation of Δu to include coefficients. We can also consider equations driven by operators of the form

$$Lu := \mathrm{Tr}(A(x) D^2 u), \tag{2}$$

which are called *non-divergence form operators.* Analogously to divergence form operators, we can consider *degenerate elliptic* and *uniformly elliptic* non-divergence form operators.

The motivation for studying non-divergence form elliptic operators comes from the nonlinear world. Let us consider an elliptic equation of the form

$$Lu = F(D^2 u) = 0,$$

where F is a nonlinear function. There is also a right notion of ellipticity for this kind of equations, that essentially boils down to asking the linearized problem to be elliptic. In other words, we can do a first order approximation of the equation around a solution u_0, and write $u = u_0 + \varepsilon v$. Then, by the chain rule we have (note that F is a function from $n \times n$ matrices to $\mathbb{R}$),

$$Lu = F(D^2 u_0 + \varepsilon D^2 v) = \varepsilon \sum_{i,j=1}^n (\partial_{ij} F)(D^2 u_0(x)) \frac{\partial^2 v}{\partial x_i x_j} + o(\varepsilon) = \varepsilon \,\mathrm{Tr}(A(x) D^2 v) + o(\varepsilon) = 0,$$

where $A(x)$ is the matrix with $(\partial_{ij} F)(D^2 u_0(x))$ as ji-th entry. Now, understanding the solutions to $\mathrm{Tr}(A(x) D^2 v) = 0$ gives insight into the solutions of the original nonlinear problem $F(D^2 u)$.

Nonlocal elliptic operators

Nonlocal elliptic operators describe diffusive processes where long-range interactions are important. In this work, we will only consider symmetric translation-invariant nonlocal operators, so in a sense they can be viewed both as *divergence form* and *non-divergence form.*

[5] The notion of solution and other technicalities are intentionally not made precise.

Let us motivate how a nonlocal diffusive operator looks like. As in the local case, assume that L is isotropic and linear. Now, our starting point is the following identity for the Laplacian: for any $u \in C^2$,

$$-\Delta u(x) = c_n \lim_{r\to 0^+} \frac{1}{r^2|B_r|}\int_{B_r}(2u(x) - u(x+y) - u(x-y))\mathrm{d}y,$$

that we can rewrite

$$-\Delta u(x) = \lim_{r\to 0^+}\int_{\mathbb{R}^n}(2u(x) - u(x+y) - u(x-y))K_r(y)\mathrm{d}y,$$

where

$$K_r = \frac{1}{r^2|B_r|}\chi_{B_r}.$$

We can understand this kernel as taking the average of the difference of the values of u around x and $u(x)$. In this form, the translation and rotation invariance of the Laplacian are apparent, and the locality comes from the fact that we are taking the limit as $r \to 0^+$.

Now, to construct a nonlocal operator, we will choose a kernel that has support in the full space and not take limits. We have[6]

$$Lu(x) := \int_{\mathbb{R}^n}(2u(x) - u(x+y) - u(x-y))K(y)\mathrm{d}y.$$

Then, by the invariance properties that we want, we need $K(y) = K(|y|)$ to be radial. Let us take $K(y) = |y|^{-p}$, where the sign of the exponent is chosen to take into account that interaction is stronger between points that are close than between points that are far.

The next step is figuring out what values of p give rise to an operator that is well defined, at least, for smooth functions of compact support. For that, we may split the integral into two pieces:

$$Lu(x) = \int_{B_1}(2u(x) - u(x+y) - u(x-y))K(y)\mathrm{d}y + \int_{\mathbb{R}^n\setminus B_1}(2u(x) - u(x+y) - u(x-y))K(y)\mathrm{d}y.$$

For the singularity, if u is at least C^2, we can use the mean value theorem to write

$$\left|\int_{B_1}(2u(x) - u(x+y) - u(x-y))K(y)\mathrm{d}y\right| \le \int_{B_1}\|D^2u\|_{L^\infty(B_1)}|y|^2K(y)\mathrm{d}y$$

so what we need is that $|y|^2K(y)$ is integrable in B_1. For the tail term, given u bounded,

$$\left|\int_{\mathbb{R}^n\setminus B_1}(2u(x) - u(x+y) - u(x-y))K(y)\mathrm{d}y\right| \le \int_{\mathbb{R}^n\setminus B_1}4\|u\|_{L^\infty(\mathbb{R}^n)}K(y)\mathrm{d}y,$$

and hence we need $K(y)$ to be integrable in $\mathbb{R}^n \setminus B_1$. Combining these properties, we obtain the Lévy condition

$$\int_{\mathbb{R}^n}\min\{1, |y|^2\}K(y)\mathrm{d}y < +\infty,$$

[6] We are using the opposite sign convention compared to local operators to be coherent with the definitions in the articles comprising this thesis.

and restricting to our example of radially decreasing powers, $K(y) = |y|^{-n-2s}$, with $s \in (0,1)$.

The model example of nonlocal operator that we deal with is the fractional Laplacian

$$(-\Delta)^s u(x) := c_{n,s} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} \mathrm{d}y.$$

As the name suggests, this operator is very closely related to the Laplacian, it satisfies the semigroup property

$$(-\Delta)^s (-\Delta)^t u = (-\Delta)^{s+t} u$$

for $s + t < 1$, and also

$$(-\Delta)^s (-\Delta)^{1-s} u = -\Delta u.$$

If we look at the Fourier side, the Fourier transform of the Laplace operator is

$$\mathcal{F}(-\Delta u) = |\xi|^2 \hat{u}(\xi),$$

and that of the fractional Laplacian is

$$\mathcal{F}((-\Delta)^s u)) = |\xi|^{2s} \hat{u}(\xi).$$

In this latter setting it is much clearer that $(-\Delta)^s$ is a pseudodifferential operator of order $2s$, and its close relation to the Laplacian.

As in the local case, there are more general classes of nonlocal elliptic operators and different notions of uniform ellipticity that are appropriate for them. In some cases, it is convenient to write

$$Lu = \int_{\mathbb{R}^n} \Big(u(x) - u(x+y) \Big) K(y) \mathrm{d}y.$$

This formulation is equivalent for kernels K that are even, but if we allow K to be asymmetric, it can encode nonlocal drift effects.

Caffarelli-Silvestre extension

The fractional Laplacian can be related to the Dirichlet-to-Neumann map for a local operator, using a trick that is known to the PDE community as the Caffarelli-Silvestre extension, because it was introduced by them in [49]. Much before that, it had been discovered in the context of probability [192, 153].

The idea of the extension is adding an extra variable so that the nonlocal behaviour can happen driven by a local operator across the extra space. Our local operator of interest acts on functions defined on $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, +\infty)$ as

$$L_s w := \Delta_x w + \frac{1-2s}{y} \partial_y w + \partial_{yy} w = y^{2s-1} \operatorname{Div}(y^{1-2s} \nabla w).$$

This operator is degenerate elliptic, and shares many properties of the Laplacian. For $s = \frac{1}{2}$, $L_{1/2}$ is exactly the Laplacian. For other values of s, the structure of L_s reminds of the spherical Laplacian, with $1 - 2s$ replacing the usual dimensional $n - 1$.

Theorem. *Let $s \in (0,1)$, $\varepsilon > 0$, and $u \in C^{2s+\varepsilon}(B_1) \cap L^1(\mathbb{R}^n)$. Let $\tilde{u} : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ be the unique solution to*

$$\begin{cases} L_s \tilde{u} &= 0 \quad \text{in} \quad \{y > 0\} \\ \tilde{u}(x,0) &= u(x) \quad \text{for} \quad x \in \mathbb{R}^n \end{cases}$$

with sublinear growth at infinity. Then,

$$(-\Delta)^s u(x) = -d_s \lim_{y \to 0^+} y^{1-2s} \partial_y \tilde{u}(x,y) \quad \text{in } B_1,$$

where $d_s > 0$.

The practical significance of this result is that it allows to extrapolate the well established theory for local elliptic equations to the less known world of integro-differential elliptic equations. Applications range from basic regularity estimates to Poisson kernel representations and monotonicity formulas.

As a final remark, the Caffarelli-Silvestre extension is not only available for the fractional Laplacian, but to the broader class of translation-invariant integro-differential operators with kernels of the form $K(y) = |Ay|^{-n-2s}$, where A is a positive definite symmetric matrix.

Common properties of solutions

Maximum and comparison principles

Consider a solution to

$$Lu = 0 \quad \text{in } \Omega,$$

where L is an elliptic operator. Then, u does not attain interior local maxima nor minima. The intuition behind this fact is that if there were a local maximum, the diffusion effects would make the value of the maximum decrease, contradicting the fact that u is a solution to the stationary-state equation. In particular, this implies that if $u \geq 0$ on $\partial\Omega$, then $u \geq 0$ in Ω.

Since the equation is linear, we can consider two solutions

$$Lu = Lv = 0 \quad \text{in } \Omega,$$

such that $u \leq v$ in $\partial\Omega$. Applying the maximum principle to $v - u$, we obtain that $u \leq v$ in Ω. This is called the comparison principle.

Maximum and comparison principles hold for a broad class of elliptic and parabolic equations, in divergence and non-divergence form, local and nonlocal... They are a key distinguishing factor of this area of PDE, and one of the basic ingredients in obtaining regularity of solutions.

Variational structure

Harmonic functions are local minimizers of the Dirichlet energy. That is, if $\Delta u = 0$ in Ω and K is compactly contained in Ω, then

$$\int_\Omega |\nabla u|^2 \leq \int_\Omega |\nabla v|^2,$$

for any C^1 function v such that $v = u$ in $\Omega \setminus K$. This fact can be interpreted as harmonic functions having the *least necessary oscillations* for any given boundary data, and therefore it yields regularity properties.

Elliptic and parabolic equations in divergence form, and in particular those for translation-invariant operators, can also be viewed as energy minimization problems, both in the local and in the nonlocal framework.

Random walks and random jumps

The Wiener process (also called Brownian motion) is a stochastic process X_t with $t \geq 0$ that models a random walk. It has the following properties:

- $X_0 = 0$ with probability 1.
- X_t has *independent increments*, that is, for all $\tau > 0$, $X_{t+\tau} - X_t$ is independent of X_s for all $s \in [0, t)$.
- $X_{t+\tau} - X_t$ has a normal distribution with mean 0 and variance τ.
- X_t is almost surely continuous in t.

The Wiener process is very related to the Laplace operator, in the following sense. Given a stochastic process, we can define its infinitesimal generator acting on functions $u : \mathbb{R}^n \to \mathbb{R}$ as

$$Lu(x) := \lim_{h\to 0} \frac{\mathbb{E}\left(u(x + X_h)\right) - u(x)}{h}.$$

In the case of the Wiener process, the infinitesimal generator is the Laplacian (recall that the Laplacian of u at x can be interpreted as the average of u near x minus $u(x)$).

The infinitesimal generator of a stochastic process encodes a lot of information about it. It can be used, for example, to compute statistics as the expected payoff or exit times. More concretely, let Ω be a bounded domain with continuous boundary, and let $\varphi : \partial\Omega \to \mathbb{R}$. Now, imagine that we start a random walk at a point $x \in \Omega$ and we collect the payoff φ the first time the process touches $\partial\Omega$. We want to know what is the expected payoff.

In other words, we want to compute

$$u(x) := \mathbb{E}(\varphi(x + X_{t^*})),$$

where t^* is the first exit time. Then, for any point x and a small time $t > 0$,

$$u(x) = \mathbb{E}(u(x + X_t)) + o(t),$$

where the error $o(t)$ is due to the small probability that the process exits Ω. Therefore,

$$\lim_{t\to 0} \frac{\mathbb{E}((u(x + X_t))) + o(t) - u(x)}{t} = 0,$$

which implies $Lu = 0$. In the case of the Wiener process, in particular, the expected payoff is a harmonic function.

Now, consider a more general stochastic process where we drop the continuity and normality assumptions, and we allow jumps instead. Under some technical conditions[7], we can carry out a similar reasoning, and we find that the infinitesimal generators of *random jump processes* are nonlocal elliptic operators.

[7] The right class of processes to consider here are Lévy processes, see [27].

Regularity theory for elliptic and parabolic PDE

A central question in the study of PDE is the following:

Given a solution to a certain PDE, will it be smooth or will it have singularities?

This question is motivated by several points of view. If we think about a PDE that comes from a physical problem, the existence of singular solutions may be related to the model not accurately representing the material reality, and then it is something to avoid. But sometimes singular solutions have obvious physical meanings, such as a fracture in a material subject to external forces. Even more, sometimes finding a singular solution to a PDE has motivated the discovery of a physical phenomenon related to it, such as the Big Bang.

The question of regularity is intimately related to the question of what is the right space to look for solutions to a PDE. As we have already pointed out, proving existence, uniqueness and stability of classical solutions can be an extremely difficult task, even when they exist, as the spaces C^k do not have good compactness properties.

As a consequence, usually the task is split into two more manageable subtasks:

- Finding a generalized solution in a bigger space.
- Proving *a posteriori* that the generalized solution is more regular, and therefore it is actually a classical solution. This is the job of regularity theory.

Finally, regularity plays a crucial role in numerical analysis. Since numerical methods can only perform a finite number of computations, they usually approximate a PDE by a discrete process, replacing derivatives with difference quotients. Then, we want to know how far the approximated solution is from the true solution. To compute these error bounds, we can use estimates on higher order derivatives of solutions.

Qualitative and quantitative estimates

Regularity results can be broadly classified into two types:

Qualitative results

Qualitative results assert that a solution to a PDE class belongs to a certain function space. For example, the smoothing effect of the heat equation can be written as:

Proposition. *Let $u(x,t)$ be a solution to $u_t = \Delta u$ in $\mathbb{R}^n \times (0,\infty)$, with initial condition $u(x,0) = u_0(x)$ in $L^\infty(\mathbb{R})$.*

Then, for any $t > 0$, $u(\cdot,t) \in C^\infty(\mathbb{R})$.

With this result, we know that the solution becomes smooth after any amount of time, but we do not know how fast or how smooth. This would be an interesting result from the point of view of physics (there is no singularity), but impractical for numerics.

Quantitative results

Quantitative results are inequalities that relate norms of solutions to a PDE. For example, the L^∞ gradient estimates for harmonic functions look as follows:

Proposition. *Let $u(x)$ be a bounded solution to $\Delta u = 0$ in the unit ball B_1. Then,*

$$\|\nabla u\|_{L^\infty(B_{1/2})} \le C\|u\|_{L^\infty(B_1)},$$

where the constant C depends only on the dimension.

With this kind of estimates we can compute error bounds and convergence rates of algorithms.
An important subclass of quantitative estimates are *a priori estimates*, which provide quantitative regularity information even when the solution is not known to exist. They take the form of

Proposition. *Let $u \in C^k$ be a solution to $\Delta u = 0$ in the unit ball B_1. Then,*

$$\|D^k u\|_{L^2(B_{1/2})} \le C_k\|u\|_{L^2(B_1)},$$

where the constant C_k depends only on k and the dimension.

These estimates are usually used to prove existence of solutions using the continuity method or a fixed point theorem.

Function spaces

Quantitative regularity results are written in terms of function norms. When studying regularity theory, it soon becomes obvious that C^k spaces are not enough.
In the following, we will recall the most paradigmatic function spaces used in regularity theory. They all are complete vector spaces with respect to their norm, i.e. Banach spaces.

Lebesgue spaces

Lebesgue spaces, L^p, are spaces of measurable (possibly singular) functions. For a nice domain $\Omega \subset \mathbb{R}^n$ (open, with Lipschitz boundary), and $p \in [1, \infty]$, we define the L^p norms on measurable functions $f : \Omega \to \mathbb{R}$, as

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f|^p\right)^{\frac{1}{p}}, \text{ for } 1 \le p < +\infty,$$
$$\|f\|_{L^\infty(\Omega)} := \sup_\Omega |f|$$

Then,

$$L^p(\Omega) := \{f : \Omega \to \mathbb{R}, \text{measurable}, \|f\|_{L^p(\Omega)} < +\infty\}.$$

Intuitively, the bigger the p is, the less singular an integrable function is allowed to be. This can be seen by the inclusion $L^q(\Omega) \subset L^p(\Omega)$ for any $q > p$, when Ω is a bounded domain, which is the relevant case in regularity theory.
In Lebesgue spaces, functions are often identified up to a zero measure set. We will follow this convention.

Hölder spaces

Hölder spaces, $C^{0,\alpha}$, are spaces of continuous functions. For a nice domain Ω, and $\alpha \in (0,1]$, we define the $C^{0,\alpha}$ seminorms (and norms) on continuous functions $f : \Omega \to \mathbb{R}$, as

$$[f]_{C^{0,\alpha}(\Omega)} := \sup_{x,y\in\Omega} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

and

$$\|f\|_{C^{0,\alpha}(\Omega)} := \|f\|_{L^\infty(\Omega)} + [f]_{C^{0,\alpha}(\Omega)}.$$

Then,

$$C^{0,\alpha}(\Omega) := \{f : \Omega \to \mathbb{R}, \text{continuous}, \|f\|_{C^{0,\alpha}(\Omega)} < +\infty\}.$$

We can also define the higher order Hölder spaces $C^{k,\alpha}$ as follows. Given $k \in \mathbb{N}$ and $\alpha \in (0,1]$, we define the higher order Hölder norm as

$$\|f\|_{C^{k,\alpha}(\Omega)} := \|f\|_{C^k(\Omega)} + [D^k f]_{C^{0,\alpha}(\Omega)},$$

where

$$\|f\|_{C^k(\Omega)} := \|f\|_{L^\infty(\Omega)} + \sum_{j=1}^{k} \|D^j f\|_{L^\infty(\Omega)}.$$

$C^{k,\alpha}$ spaces interpolate C^k spaces, in the sense that if $0 < \alpha < \beta < 1$,

$$C^0 \supset C^{0,\alpha} \supset C^{0,\beta} \supset C^{0,1} \supset C^1 \supset C^{1,\alpha} \supset C^{1,\beta} \supset C^{1,1} \supset \dots$$

Weak derivatives

To make sense of solutions to a PDE that are less regular than what is needed to be evaluated pointwise, we need to give a new meaning to differentiation. This is done through integration by parts.

For example, if $u \in C^1(\mathbb{R})$, we have that for any $\eta \in C^1(\mathbb{R})$ with compact support,

$$\int_{\mathbb{R}} u'(x)\eta(x)\mathrm{d}x = -\int_{\mathbb{R}} u(x)\eta'(x)\mathrm{d}x.$$

Then, we may identify u' with the mapping

$$\eta \to -\int_{\mathbb{R}} u(x)\eta'(x)\mathrm{d}x.$$

Now let us consider $v \in L^1(\mathbb{R})$, and assume that

$$\eta \to -\int_{\mathbb{R}} v(x)\eta'(x)\mathrm{d}x$$

is well defined. If there exists a locally integrable function w such that

$$\int_{\mathbb{R}} w(x)\eta(x)\mathrm{d}x = -\int_{\mathbb{R}} v(x)\eta'(x)\mathrm{d}x,$$

we say that w is the weak derivative of v and we write $w = v'$. For example, the weak derivative of $v(x) = |x|$ is

$$v'(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x > 0. \end{cases}$$

Note that defining $v'(0)$ is superfluous, as we are considering locally integrable functions, and, as usual, we identify functions that are equal except for a set of zero measure.

Weak differentiation can be extended to higher dimensions and higher orders by integrating by parts more times in the exact same way, and all the usual properties of derivatives also hold. Moreover, when derivatives exist in a classical sense, they coincide with weak derivatives.

Sobolev spaces

Sobolev spaces, $W^{k,p}$, are spaces of measurable functions with weak derivatives. For a nice domain Ω, $k \in \mathbb{N}$, and $p \in [1, \infty]$, we define the $W^{k,p}$ norms on measurable functions $f : \Omega \to \mathbb{R}$, as

$$\|f\|_{W^{k,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \sum_{j=1}^{k} \|D^j f\|_{L^p(\Omega)},$$

where $D^j f$ has to be understood in the weak sense. Then,

$$W^{k,p}(\Omega) := \{f \in L^p(\Omega) : \|f\|_{W^{k,p}(\Omega)} < +\infty\}.$$

Now, we have two indices to control how singular the functions in $W^{k,p}$ can be: differentiability (k) and integrability (p). As both numbers increase, the functions become less singular.

When $p = 2$, the spaces $W^{k,2}$ become Hilbert spaces, and they are also denoted H^k. Moreover, one can define $W_0^{k,p}$ as the closure of C_c^∞ with respect to the $W^{k,p}$ norm. This space can be understood as $W^{k,p}$ functions with zero boundary conditions, where the boundary conditions are also seen in the Sobolev sense.

Lipschitz functions

There are two natural ways of thinking about Lipschitz functions. On the one hand, we can think of functions that belong to the space $C^{0,1}$ as a Hölder space.

On the other hand, we can think of functions that admit one bounded weak derivative, that is, the Sobolev space $W^{1,\infty}$.

The spaces $C^{0,1}$ and $W^{1,\infty}$ coincide when the domain is the full space or has a smooth enough boundary. The key fact in this equivalence is Rademacher's theorem, that states that Lipschitz functions are differentiable almost everywhere.

Embeddings

Sobolev, Hölder and Lebesgue spaces are related, and there are inclusions between them. When $p < n$, we have the Gagliardo-Nirenberg-Sobolev inequality.

Theorem. *Let $p < n$ and $u \in C_c^1(\mathbb{R}^n)$. Then,*

$$\|u\|_{L^{p^*}} \leq C(n,p)\|\nabla u\|_{L^p(\mathbb{R}^n)},$$

where

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

This can be understood as $W^{1,p} \subset L^{p^*}$ when $p < n$. On the other hand, when $p > n$, we have the Morrey inequality.

Theorem. *Let $p > n$ and $u \in C^1_c(\mathbb{R}^n)$. Then,*

$$\|u\|_{C^{0,\alpha}} \leq C(n,p)\|\nabla u\|_{L^p(\mathbb{R}^n)},$$

where

$$\alpha = 1 - \frac{n}{p}.$$

Similarly, this implies that $W^{1,p} \subset C^{0,\alpha}$ when $p > n$. We purposely do not consider the case $p = n$ for simplicity. Keep in mind that, for compactly supported functions and all $p < q$,

$$\|u\|_{L^p(B_R)} \leq C(n,p,q)\|u\|_{L^q(B_R)},$$

and we may use this fact to obtain that $W^{1,n} \subset L^p$ for all $p < \infty$, for example.

These inequalities can be combined and iterated to give the following.

Theorem. *Let $u \in C^\infty_c(\mathbb{R}^n)$. Then, let $k \in \mathbb{N}$ and $p \geq 1$. Then,*

a) If $kp < n$, then

$$\|u\|_{L^{p_k}(\mathbb{R}^n)} \leq C_1(n,p,k)\|Du\|_{L^{p_{k-1}}(\mathbb{R}^n)} \leq \ldots \leq C_k(n,p,k)\|D^k u\|_{L^{p_0}(\mathbb{R}^n)},$$

where

$$\frac{1}{p_j} = \frac{1}{p} - \frac{j}{n}.$$

b) If $kp > n$, and n/p is not an integer, let $m = \lfloor n/p \rfloor$. Then,

$$\|u\|_{C^{k-m-1,\alpha}(\mathbb{R}^n)} \leq C_m(n,p,k)\|D^{k-m}u\|_{L^{p_m}(\mathbb{R}^n)} \leq \ldots \leq C_k(n,p,k)\|D^k u\|_{L^{p_0}(\mathbb{R}^n)},$$

where

$$\alpha = 1 - \frac{n}{p_m} = m + 1 - \frac{n}{p}.$$

There are also similar versions of the embeddings for bounded domains.

Linear theory

For harmonic functions, one can deduce interior regularity directly from the Poisson kernel representation. When we move to the broader class of linear elliptic PDE, such an explicit formula fails to exist and we need other tools.

In this section, we will consider equations in divergence form,

$$Lu := \mathrm{Div}(A(x)\nabla u) = f,$$

and in non-divergence form,

$$Lu := \mathrm{Tr}(A(x)D^2 u) = f,$$

where the matrix $A(x)$ is symmetric, and uniformly elliptic, i.e.

$$\lambda I \leq A(x) \leq \Lambda I.$$

To fix ideas, let us consider first the Poisson equation $\Delta u = f$. Then, it is clear that if $u \in C^{k+2}$, $f \in C^k$. The converse, however, is false. There are examples with continuous f such that $u \notin C^2$, not even in $C^{1,1}$, and such a simple result also fails when one considers $f \in L^\infty$: it is not true that then D^2u is also bounded.

One expects sharp results to say that D^2u and f belong to the same regularity class, since it is trivial that f is *at least as regular* as D^2u. But we cannot hope to prove such results in C^k or $C^{k,1}$ spaces. This is why we have to move to the finer Hölder and Sobolev spaces.

For more general equations, we also expect to *gain two derivatives* with respect to the right-hand side, for regular enough coefficients.

Schauder estimates

Schauder estimates are a priori regularity estimates in Hölder spaces. There is the following version for equations in non-divergence form.

Theorem. *Let $k \in \mathbb{N}$, $\alpha \in (0,1)$, and let $u \in C^{k+2,\alpha}$ be a solution to*

$$\mathrm{Tr}(A(x)D^2u) = f \quad \text{in } B_1,$$

with $A, f \in C^{k,\alpha}$, and $A(x)$ uniformly elliptic. Then,

$$\|u\|_{C^{k+2,\alpha}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{k,\alpha}(B_1)}\right),$$

where C depends only on k, α, $\|A(x)\|_{C^{k,\alpha}(B_1)}$, the dimension and ellipticity constants.

And also the version for equations in divergence form.

Theorem. *Let $k \in \mathbb{N}$, $k \geq 1$, $\alpha \in (0,1)$, and let $u \in C^{k+1,\alpha}$ be a solution to*

$$\mathrm{Div}(A(x)\nabla u) = f \quad \text{in } B_1,$$

with $A, f \in C^{k,\alpha}$, and $A(x)$ uniformly elliptic. Then,

$$\|u\|_{C^{k+1,\alpha}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{k-1,\alpha}(B_1)}\right),$$

where C depends only on k, α, $\|A(x)\|_{C^{k,\alpha}(B_1)}$, the dimension and ellipticity constants.

The case $k = 0$ can be adapted substituting the $C^{k-1,\alpha}$ norm by a L^q norm with $q = n/(1-\alpha)$, that is, the Lebesgue space with the scaling that $C^{-1,\alpha}$ would have[8].

[8] One can also consider the space $C^{-1,\alpha}$ as the (distributional) derivatives of C^α functions. In this case a version of Schauder estimates also hold, see [112, Theorem 8.33]

Calderón-Zygmund estimates

Calderón-Zygmund estimates are a priori estimates in Sobolev spaces. We will only state them for the Laplacian.

Theorem. *Let $k \in \mathbb{N}$, $p \in (1, \infty)$, and let $u \in W^{k+2,p}$ be a solution to*

$$\Delta u = f \quad \text{in } B_1,$$

with $f \in W^{k,p}$. Then,

$$\|u\|_{W^{k+2,p}(B_{1/2})} \leq C \left(\|u\|_{L^p(B_1)} + \|f\|_{W^{k,p}(B_1)} \right),$$

where C depends only on k, p, and the dimension.

Nonlinear theory

A central idea in the regularity theory for nonlinear PDE is obtaining estimates for the associated linearized problem that do not depend on the regularity of the coefficients. Then, one can proceed as follows. Take a nonlinear PDE of the form

$$F(D^2 u) = f$$

and differentiate it in the direction e. Then, at least formally,

$$\sum_{i,j=1}^{n} \partial_{ij} F(D^2 u) \partial_{ij} u_e = f_e,$$

that is, u_e solves a linear PDE with coefficients that depend on u itself, and hence we cannot assume any regularity of the coefficients a priori.

The Cacciopoli energy inequality

Energy methods are a very powerful tool to obtain initial regularity for divergence structure equations, and are inspired by the elementary fact that if $\Delta u = 0$, then $|\nabla u|^2$ is subharmonic, and then it is bounded pointwise by its average, which is a H^1-to-Lipschitz kind of estimate for u.

For more general equations, the picture is not so simple, but we have the following.

Theorem. *Let $u \in H^1(B_1)$ be a nonnegative weak solution to $Lu \geq 0$ in B_1, where L is a divergence form uniformly elliptic operator. Then, for any $\eta \in C_c^1(B_1)$,*

$$\int_{B_1} |\nabla(\eta u)|^2 \leq C \|\nabla \eta\|^2_{L^\infty(B_1)} \int_{\operatorname{supp} \eta} u^2.$$

If we now choose $\eta \in C_c^\infty(B_1)$ to be a cutoff function with $\eta \equiv 1$ in $B_{1/2}$, we deduce the reverse Sobolev inequality

$$\int_{B_{1/2}} |\nabla u|^2 \leq C \int_{B_1} u^2,$$

that encodes regularity of u.

Harnack's inequality and Hölder regularity

Harnack's inequality is one of the most fundamental estimates for elliptic equations, and it can be understood as a quantitative form of the maximum principle.

Theorem. *Let $u \geq 0$ be a solution to $Lu = 0$ in B_1, with L uniformly elliptic, either in divergence or non-divergence form. Then,*

$$\sup_{B_{1/2}} u \leq C \inf_{B_{1/2}} u,$$

where C depends only on the dimension and ellipticity constants.

Now, by applying Harnack's inequality to $u - \inf u$ and $\sup u - u$, the following oscillation decay follows.

Corollary. *Let $u \in L^\infty$ be a solution to $Lu = 0$ in B_1, with L uniformly elliptic, either in divergence or non-divergence form. Then,*

$$\sup_{B_{1/2}} u - \inf_{B_{1/2}} u \leq (1-c)\left(\sup_{B_1} u - \inf_{B_1} u\right),$$

where $c > 0$ depends only on the dimension and ellipticity constants.

Iterating the oscillation decay, it follows that

$$\sup_{B_{2^{-k}}} u - \inf_{B_{2^{-k}}} u \leq C(1-c)^k,$$

which implies $u \in C^\alpha$ for some $\alpha > 0$.

The key point here is that bounded solutions to uniformly elliptic equations are C^α without assuming any regularity on the coefficients. This makes this result useful for nonlinear equations. It is worth mentioning that all these results are also known for elliptic equations with right-hand side.

Hilbert's XIX problem

Dating from 1900, Hilbert's XIX problem asked whether solutions to *regular variational problems* are always analytic, motivated by the fact that the Laplace equation, the minimal surface equation and some others were known to admit only analytic solutions. The problem was solved independently in the 50s by De Giorgi and Nash.

In modern language, we consider solutions to the minimization problem

$$\min E(u) := \int_\Omega \mathcal{L}(\nabla u),$$

where the Lagrangian $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}$ is analytic and uniformly convex. Then, we ask if all local minimizers of $E(u)$ are analytic.

The natural space to study this question is H^1, where one can prove existence and uniqueness of minimizers under suitable boundary conditions. Then, by the minimality, for any $\varphi \in H_0^1(\Omega)$,

$$0 = \frac{\partial}{\partial \varepsilon} E(u + \varepsilon\varphi) = \int_\Omega D\mathcal{L}(\nabla u)\nabla\varphi = -\int_\Omega \mathrm{Div}(\mathrm{D}\mathcal{L}(\nabla \mathrm{u}))\varphi,$$

and therefore the minimizers are weak solutions of

$$\mathrm{Div}\,(D\mathcal{L}(\nabla u)) = 0.$$

Hence, differentiating the equation (formally) we get

$$\mathrm{Div}\left(D^2\mathcal{L}(\nabla u)\nabla u_e\right) = 0,$$

a divergence form equation, that is uniformly elliptic because $\mathcal{L}$ is uniformly convex.

The key point to prove regularity of minimizers of $E(u)$ is to obtain some initial regularity, that is, prove that given a solution $u \in H^1$, actually $u \in C^{1,\alpha}$. Then, one can use Schauder estimates to further this regularity to C^∞. To do that, we can split the argument into two parts:

- Using the Cacciopoli and Sobolev inequalities, prove that u_e is bounded.
- Then, by an oscillation decay argument, $u_e \in C^\alpha$.

Boundary regularity

For the sake of simplicity we will consider the Dirichlet problem for the Laplacian with homogeneous boundary conditions. The simplest case, a flat boundary, can be assimilated to the interior regularity theory by an odd reflection. Indeed, if $\Delta u = f$ in $B_1 \cap \{x_n > 0\}$ and $u = 0$ on $\{x_n = 0\}$, we can consider, writing $x = (x', x_n)$ in $\mathbb{R}^{n-1} \times \mathbb{R}$,

$$\tilde{u}(x', x_n) = \mathrm{sgn}(x_n)u(x', |x_n|),$$

and then $\Delta\tilde{u} = \tilde{f}$ in B_1, so we can apply interior Schauder or Calderón-Zygmund estimates to $\tilde{u}$ to deduce the boundary regularity of u. In short, we gain two derivatives as in interior estimates.

Now let us consider a solution to $\Delta u = f$ in a general domain Ω, and consider a diffeomorphism $\phi : \Omega \to \tilde{\Omega}$ that sends the boundary of Ω to $\{x_n = 0\}$. Then, since u is a weak solution, it solves

$$\int_\Omega \nabla u \cdot \nabla\eta \mathrm{d}x = \int_\Omega f\eta \mathrm{d}x,$$

which after the change of variables becomes

$$\int_{\tilde{\Omega}} \nabla u^\top (D\phi)^\top D\phi\nabla\eta \mathrm{d}y = \int_{\tilde{\Omega}} D\phi\nabla u \cdot D\phi\nabla\eta \mathrm{d}y = \int_{\tilde{\Omega}} f\eta \mathrm{d}y,$$

choosing $|D\phi| = 1$ without loss of generality, which is the weak form of a divergence form equation with coefficients $A(x) = (D\phi)^\top(D\phi)$. The important point here is that the regularity of the coefficients depends on the regularity of the boundary, in the sense that if the boundary is $C^{k,\alpha}$ for $k \geq 1$, $A(x) \in C^{k-1,\alpha}$, and then by Schauder estimates $u \in C^{k,\alpha}$.

In general, we conclude that if a domain is of class $C^{k,\alpha}$, then harmonic functions that vanish on the boundary are $C^{k,\alpha}$ up to the boundary. However, the situation changes drastically when we lower the regularity to Lipschitz. Then, harmonic functions are only C^α, with an exponent that depends on the Lipschitz constant. This is due to the fact that, when we zoom in, C^1 domains *improve*: they become flatter and converge to a half-space, but Lipschitz domains look the same at all scales.

Free boundary problems

Free boundary problems are a particular class of overdetermined problems where the boundary of the domain is an unknown that provides the missing degrees of freedom. They can be seen as the counterpart to the study of boundary regularity. For example, we know that if a domain is $C^{k,\alpha}$, then harmonic functions with smooth Dirichlet conditions are $C^{k,\alpha}$ up to the boundary. We might ask:

If Ω is a Lipschitz domain such that the Poisson kernel is $C^{k,\alpha}$, is it true that the boundary is locally $C^{k,\alpha}$?

The answer is affirmative, see [47, 196].

Another angle to free boundary problems is from the physical point of view. If we want to model a system with more than one phase, and in each phase a different PDE describes the behaviour of a physical quantity u, we may study the problem where we do not know a priori what space is occupied by each phase, but instead we know some boundary conditions about how they interact. For example, if two phases are represented by positive and negative values of u, we could have something that looks like

$$\begin{cases} F_1(u) & = & 0 & \text{in} & \Omega \cap \{u > 0\} \\ F_2(u) & = & 0 & \text{in} & \Omega \cap \{u < 0\} \\ G(u_+, u_-) & = & 0 & \text{on} & \Omega \cap \{u = 0\} \\ u & = & h & \text{on} & \partial\Omega, \end{cases}$$

where F_1 and F_2 represent the PDE that each phase solves, G is an interaction law and h is a boundary condition. Most of this Thesis is devoted to the study of the regularity theory for elliptic and parabolic free boundary problems.

Part I

Regularity theory for obstacle problems

Beware of truths, for they hide nothing but conventions.[9]

[9] Paraphrasing the graduation speech of Cassius Manuel Pérez de los Cobos Hermosa, Universitat Politècnica de Catalunya (2015).

Introduction to Part I

Obstacle problems are an archetypal example of free boundary problem consisting of a nonlinear elliptic or parabolic PDE of the form

$$\min\{Lu, u-\varphi\} = 0 \quad \text{or} \quad \min\{u_t - Lu, u-\varphi\} = 0.$$

In these problems, the domain is divided into two unknown sets, the contact set $\{u = \varphi\}$, where the solution u touches the obstacle φ, and the set $\{u > \varphi\}$, where u satisfies the PDE. As we already mentioned, the most important questions in regularity theory concern the regularity of u *across* the free boundary $\partial\{u > \varphi\}$, and the structure and regularity of $\partial\{u > \varphi\}$.

The obstacle problem

Let us consider an elastic membrane attached to a wire, described as the graph of a function $u : \Omega \to \mathbb{R}$ with prescribed values $u|_{\partial\Omega} = g$ on the wire.

Now, if this membrane is in equilibrium, it minimizes its energy. In our model, we will consider two contributions to the energy: elastic and gravitational. Thence we have

$$\mathcal{E}(u) = \mathcal{E}_e(u) + \mathcal{E}_g(u) = \int_\Omega \sqrt{1 + |\nabla u|^2} + u,$$

normalizing to 1 all the relevant physical constants. To understand the equilibrium state of the membrane we would like to find u that minimizes $\mathcal{E}(u)$. But this is a nonlinear problem, and nonlinear problems are hard.

The first reasonable step towards solving the nonlinear problem is to attack its linearized version. If $|\nabla u|$ is small[10], then we can do the first order approximation

$$\mathcal{E}(u) \approx \int_\Omega 1 + \frac{|\nabla u|^2}{2} + u,$$

and then look for minimizers of

$$E(u) = \int_\Omega \frac{|\nabla u|^2}{2} + u.$$

Now, if u is a minimizer of $E(u)$, we will have that for all η smooth and compactly supported in Ω, $E(u+\eta) \geq E(u)$. In particular, this implies (at least formally) that

$$0 = \frac{\partial}{\partial\varepsilon} E(u + \varepsilon\eta) = \int_\Omega \nabla u \cdot \nabla\eta + \eta = \int_\Omega (-\Delta u + 1)\eta,$$

[10] In this case, this is not a reasonable physical assumption.

and since this holds for every η, $\Delta u = 1$, which is the Euler-Lagrange equation associated to the functional $E(u)$.

Let us extrapolate the reasoning to the case where we have the same membrane attached to a wire, but there is also an obstacle below, described as the graph oh $\varphi : \Omega \to \mathbb{R}$. Then, we consider the minimization problem

$$\min E(u) : u \geq \varphi.$$

By a similar reasoning, we deduce that at points x where $u(x) > \varphi(x)$, $\Delta u(x) = 1$. But when $u(x) = \varphi(x)$, then we are only allowed to perturb u away from the obstacle, and then we obtain something like

$$\int_\Omega (-\Delta u + 1)\eta \geq 0,$$

where $\eta \geq 0$ is supported near x, and then $\Delta u \leq 1$. Note that *a priori* we do not know at which points x we have $u(x) = \varphi(x)$ or $u(x) > \varphi(x)$. The obstacle problem can be summarized in the set of equations

$$\begin{cases} u & \geq & \varphi & \text{in} & \Omega \\ \Delta u & \leq & 1 & \text{in} & \Omega \\ \Delta u & = & 1 & \text{in} & \{u > \varphi\}. \end{cases}$$

We can also look for the PDE satisfied by $v = u - \varphi$. In this case,

$$\begin{cases} v & \geq & 0 & \text{in} & \Omega \\ \Delta v & \leq & f & \text{in} & \Omega \\ \Delta v & = & f & \text{in} & \{v > 0\}, \end{cases}$$

with $f = 1 - \Delta\varphi$.

Then, we can combine the equations in the form

$$\Delta v = f\chi_{\{v>0\}},$$

which has the advantage of not having a constraint anymore, but is nonlinear. It is usual to impose the condition $f \geq c_0 > 0$.[11] Since the right-hand side is bounded, we obtain from Calderón-Zygmund estimates that $u \in C^{1,\alpha}$, and in particular ∇u is well defined and vanishes on the free boundary.

Using this fact, the problem can be rewritten to highlight the importance of the free boundary. If we write $\Gamma = \partial\{u > 0\} \cap \Omega$, then u solves

$$\begin{cases} \Delta u & = & f & \text{in} & \{u > 0\} \\ u & = & 0 & \text{on} & \Gamma \\ \nabla u & = & 0 & \text{on} & \Gamma \end{cases}$$

Having both Dirichlet and Neumann conditions would be an overdetermined problem. But in this case, the domain is also an unknown, and the problem has a unique solution.

[11] If the obstacle does not oscillate too much, this is a reasonable physical assumption. Moreover, without this assumption the contact set can be any closed set, cf. [96, Proposition 5.2].

Motivations and applications

Since many phenomena can be described by diffusion, it is not surprising that the same kinds of equations appear again and again in different settings. Here we describe some examples, and we point out the interested reader to the book [160] and the survey papers [36, 100, 165] for more.

Physics: Stefan problem

Probably the oldest studied free boundary problem, the Stefan problem dates back to the 19th century. It describes the evolution of a phase transition from solid to liquid, in which the solid is kept at the melting point, and the evolution of temperature in the liquid is described by the heat equation.

In its formulation in physical variables, $\theta(x,t)$ represents the temperature of a mixture of ice and water. Let Ω be a bounded domain, $\theta_0(x) \geq 0$ the initial temperature, and $g : \partial\Omega \to [0,\infty)$ the boundary condition. Then, we want to find the evolution of $\theta(x,t)$ in time. We set $\theta(\cdot,0) = \theta_0$ and $\theta(x,t) = g(x)$ for all $(x,t) \in \partial\Omega \times (0,\infty)$.

The set $\{\theta = 0\}$ represents the ice, while the set $\{\theta > 0\}$ represents the water. In the water, the temperature follows the heat equation

$$\theta_t = \Delta\theta \quad \text{in } \{\theta > 0\}.$$

To see how the free boundary $\partial\{\theta > 0\}$ evolves, we do the following physical reasoning. The speed of melting must be proportional to the speed of heat transfer at the boundary. Moreover, the speed at which the boundary moves is proportional to the speed of melting, and the speed of heat transfer is proportional to the temperature gradient. Therefore, we expect something like

$$\frac{\theta_t}{|\nabla\theta|} = v = |\nabla\theta|,$$

where v is the speed at which the free boundary moves, and the first equality comes from the fact that $\theta = 0$ is constant on the moving free boundary. The condition $\theta_t = |\nabla\theta|^2$ on $\partial\{\theta > 0\}$ is called the Stefan condition, and together with the heat equation in the liquid phase, completely describes the evolution of θ.

The Stefan problem is related to the obstacle problem thanks to the change of variables

$$u(x,t) = \int_0^t \theta(x,s)\mathrm{d}s,$$

called Duvaut transformation. Then, this new function solves the parabolic obstacle problem

$$\begin{cases} u_t - \Delta u &= \chi_{\{u>0\}} \\ u &\geq 0 \\ u_t &\geq 0. \end{cases}$$

Moreover, since $\theta \geq 0$, we have $\{u = 0\} = \{\theta = 0\}$, and the free boundaries also coincide. This allows us to study the free boundary in the Stefan problem by studying the parabolic obstacle problem with $u_t \geq 0$.

Finance: Optimal stopping

In mathematical finance, the optimal stopping problem is the problem of deciding when to stop a stochastic process to obtain the maximum payoff.

Imagine that there is a contract that allows us to sell a commodity for a price $\varphi : \mathbb{R}^n \to \mathbb{R}$ that depends on the state of the market at any given time. What is the rational price for this contract?

If we represent the state of the market by the stochastic process $x + X_t$, where x is the initial state and X_t is the Wiener process, and we call $u(x)$ to this price, we have

$$u(x) = \max_{t^*} \mathbb{E}(\varphi(x + X_{t^*})),$$

where we take the maximum among all possible stopping times. Note that t^* is also a random variable, see [83] for details.

Now, we can do the following reasoning. If we are at x, we can either sell or wait. Since we can always sell right away, the contract must be worth more than the payoff at this moment. Hence we write $u \geq \varphi$.

On the other hand, the contract must be also worth at least the same as a contract that would allow us to sell at any point in time after waiting a fixed amount $\delta > 0$. We can write this as

$$u(x) \geq \max_{s^*} \mathbb{E}(\varphi(x + X_{\delta+s^*})) = \max_{s^*} \mathbb{E}(\varphi(x + X_\delta + X_{s^*})) = \mathbb{E}(u(x + X_\delta)).$$

Now, taking the limit when $\delta \to 0$, we obtain that $Lu \leq 0$, where L is the infinitesimal generator of the process, that in the case of Brownian motion is the Laplacian. Hence $\Delta u \leq 0$.

Moreover, since we are doing an optimal strategy, in the region where $u > \varphi$ we are forced to wait for a positive amount of time, and then, by the same reasoning as above, $\Delta u = 0$ in $\{u > \varphi\}$.

Putting everything together, we have that u is a solution to

$$\begin{cases} u & \geq & \varphi & \text{in} & \mathbb{R}^n \\ \Delta u & \leq & 0 & \text{in} & \mathbb{R}^n \\ \Delta u & = & 0 & \text{in} & \{u > \varphi\}, \end{cases}$$

which corresponds to the obstacle problem.

Physics and biology: Interacting particle systems

Large systems of interacting particles appear in physics (electrons), biology (individuals of a species) and material sciences, among others.

In those models, the particles repel each other when they are close, and they either attract each other when they are far, or there is a confining potential that forces them to be in a bounded region. For example, in the case of electrons in a potential well, the equilibrium configuration corresponds to the electrons being uniformly distributed in a region of space. Here, the electric potential generated by the electrons solves an obstacle problem with an obstacle that depends on the external potential, and the region where they accumulate corresponds to the contact set.

Engineering: Dam problem

The dam problem describes the filtration of water across a thick dam separating two water reservoirs with different heights. Inside the dam, there is a dry part, and a wet part where water flows across. In this problem, the free boundary is the interface between the dry and the wet parts, and the quantity that solves the obstacle problem is the integral of the pressure in the vertical direction.

Finance: American options

An American option is a contract that lets the holder buy a stock for a fixed price before a deadline $T > 0$. Thus, computing the rational price for this contract is an optimal stopping problem with a deadline. By a similar reasoning to our optimal stopping example, we find that this price $u(x, t)$ solves an obstacle problem for a backwards-in-time diffusion equation. Note that in this example, the presence of a deadline makes the problem *non-equilibrium*, and thus parabolic, in contrast to the optimal stopping problem with no deadline, that is elliptic.

Regularity theory and generalizations

In free boundary problems, the main regularity questions are:

- *What is the optimal regularity of solutions?*
- *What is the regularity of the free boundary?*

In general, understanding the regularity of the free boundary, which is just the zero set of a function, is a much harder problem than determining the regularity of solutions. However, in some cases the answers to both questions are intertwined. For example, to obtain the optimal regularity for the solutions of the thin obstacle problem, a careful study of the free boundary is needed.

The classical obstacle problem

In the obstacle problem, the regularity of solutions was studied in the 60s and early 70s in [145, 156, 109, 31], where it was shown that solutions are $C^{1,1}$. Then, the first general result for free boundaries was proved by Kinderlehrer and Nirenberg [130], who showed that if the free boundary is C^1, it is C^∞ by a perturbative argument. The gap was closed by Caffarelli in his breakthrough work of 1977, [34], where he proved that the free boundary splits into regular points, where it is $C^{1,\alpha}$, and singular points, where the contact set has zero density.

Existence and uniqueness

Existence and uniqueness of solutions follows by the direct method of the calculus of variations, using the fact that the solution to the obstacle problem

$$\begin{cases} u & \geq & \varphi & \text{in} & \Omega \\ \Delta u & \leq & 1 & \text{in} & \Omega \\ \Delta u & = & 1 & \text{in} & \{u > \varphi\}. \end{cases}$$

is the minimizer of the convex functional

$$\int \frac{|\nabla u|^2}{2} + u$$

over the closed set

$$\{v \in H^1(\Omega) : v \geq \varphi,\ v|_{\partial\Omega} = g\}.$$

A similar reasoning also works for the different reformulations of the problem, and it is standard to show that they are all equivalent. An alternative approach, that we will not discuss here, can be accomplished using viscosity solutions.

Optimal regularity

From now on, since regularity is a local property, we will focus on solutions defined in the unit ball B_1, with the formulation

$$\begin{cases} \Delta u &= f\chi_{\{u>0\}} \quad \text{in} \quad B_1 \\ u &\geq 0 \quad \text{in} \quad B_1 \\ u &= g \quad \text{on} \quad \partial B_1, \end{cases} \tag{3}$$

with $f \geq c_0 > 0$. We will assume that f is smooth for simplicity.

First, it is clear that $u \notin C^2$ because Δu is discontinuous across the free boundary. Then, by Calderón-Zygmund estimates, since the right-hand side is bounded, it belongs to L^p, and then $u \in W^{2,p}$ for all $p \geq 1$, which embeds into $C^{1,\alpha}$ for all $\alpha \in (0,1)$. In fact, one can say a bit more: the optimal regularity of the solutions is $C^{1,1}$, see for example [97].

Theorem. *Let u be a solution to the obstacle problem (3). Then,*

$$\|u\|_{C^{1,1}(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,1}(B_1)}),$$

where C depends only on the dimension.

The first step to prove the optimal regularity is obtaining the following upper bound on the growth of solutions near the free boundary, which is done by a comparison argument with a barrier.

Lemma. *Let u be a solution to the obstacle problem (3). Then, for any free boundary point $x_0 \in \partial\{u > 0\} \cap B_{1/2}$, and all $r \in (0, \frac{1}{4})$,*

$$\sup_{B_r(x_0)} u \leq Cr^2,$$

where C depends only on the dimension and $\|f\|_{C^{0,1}(B_1)}$.

Then, the proof goes roughly as follows. For any point $x \in \{u > 0\}$, let x_0 be the closest free boundary point, and let $r = |x - x_0|$. Then,

$$\|D^2u\|_{L^\infty(B_{r/4}(x))} \leq Cr^{-2}\|u\|_{L^\infty(B_{r/2}(x))} \leq Cr^{-2}\|u\|_{L^\infty(B_r(x_0))} \leq C,$$

and then $D^2u \in L^\infty$ as we wanted.

Nondegeneracy

The counterpart to the growth upper bound that gives the optimal regularity is the following lower bound, that is also obtained with a barrier argument.

Proposition. *Let u be a solution to the obstacle problem (3). Then, for any free boundary point $x_0 \in \partial\{u > 0\} \cap B_{1/2}$, and all $r \in (0, \frac{1}{4})$,*

$$\sup_{B_r(x_0)} u \geq cc_0 r^2 > 0,$$

where c depends only on the dimension.

Here is where the assumption $f \geq c_0$ becomes important. Combining the nondegeneracy with the optimal regularity, we obtain that $\|u\|_{L^\infty(B_r(x_0))}$ is comparable to r^2. That will allow us to perform blow-ups.

Blow-ups

Blow-ups are an idea that comes from minimal surfaces, that consists in zooming in at a point of a set and taking the limit. If the set is a smooth surface, the limit will be flat. The idea is to see that, when the limit is flat, the set had originally some kind of smoothness. In the setting of the obstacle problem, let $x_0 \in B_{1/2}$ be a free boundary point, and define

$$u_r(x) := \frac{u(x_0 + rx)}{r^2}.$$

By the optimal regularity, $\{u_r\}$ is bounded in $C^{1,1}$, and then by Arzelà-Ascoli we can extract a subsequence such that $u_{r_k} \to u_0$ in $C^{1,\alpha}$. Moreover, by the nondegeneracy, we have that $\|u_r\|_{L^\infty(B_1)} \geq cc_0$, and therefore $\|u_0\| \geq cc_0$. The function u_0 is called *a* blow-up of u at the point x_0. Note that the blow-up does not need to be unique.

Classification of blow-ups

By construction, a blow-up u_0 is a global solution to the obstacle problem

$$\Delta u_0 = f(x_0)\chi_{\{u_0>0\}}.$$

Let us assume $f(x_0) = 1$. Furthermore, besides freezing the right-hand side by zooming in, blow-ups satisfy two extra conditions that are crucial in classifying them: *convexity* and *homogeneity*.

The convexity is based in the following idea: D^2u_0 is harmonic in $\{u_0 > 0\}$ and 0-homogeneous. Moreover, $D^2u_0 \geq 0$ in $\{u_0 = 0\}$ because $u_0 \geq 0$. Then, by the maximum principle, D^2u_0 must be nonnegative everywhere. Proving that u_0 is homogeneous of degree 2 is more involved, and the usual proofs rely on the Weiss monotonicity formula.

Thanks to convexity and homogeneity, one can arrive at the following classification result.

Theorem. *Let u be a solution to the obstacle problem (3), and let u_0 be a blow-up of u at $x_0 \in \partial\{u > 0\} \cap B_{1/2}$. Then,*

- *Either*

$$u_0 = \frac{1}{2}(x \cdot e)_+^2,$$

  for some unit vector e.

- *Or*
$$u_0 = \frac{1}{2} x^\top A x,$$
for some positive semidefinite matrix A with $\operatorname{Tr} A = 1$.

Regular and singular points

We say that a point x_0 in the free boundary is *regular* if there is a blow-up sequence that yields a blow-up of the first type. Otherwise we say the point is *singular*. Furthermore, we have the following characterization of regular and singular points.

Proposition. *Let u be a solution to the obstacle problem* (3)*. Then, a free boundary point $x_0 \in \partial\{u > 0\} \cap B_{1/2}$ is regular if and only if*

$$\limsup_{r \to 0} \frac{|\{u = 0\} \cap B_r(x_0)|}{|B_r|} > 0,$$

and singular if and only if the contact set has density zero.

This result, combined with the classification of blow-ups, is often referred to as the Caffarelli dichotomy.

Regular points

At regular points, the blow-up has a half-space as contact set, and a hyperplane as free boundary. Using this information, one can arrive at the following.

Theorem. *Let u be a solution to the obstacle problem* (3)*. Then, the set of regular free boundary points is an open subset of $\partial\{u > 0\}$, and it is locally a C^∞ manifold.*

The proof of this fact can be organized in three steps:

- Proving that the free boundary is a flat Lipschitz graph.
- Flat Lipschitz implies $C^{1,\alpha}$.
- Bootstrapping from $C^{1,\alpha}$ to C^∞.

Here we will explain in more detail the first step. The rest can be done using the boundary Harnack and higher order boundary Harnack, see Introduction to Part II.

The key observation here is that, since u_r converges to $u_0 = \frac{1}{2}(x \cdot e)_+^2$ in $C^{1,\alpha}$ norm, at small scales

$$\|\partial_\tau u_r(x) - (\tau \cdot e)(x \cdot e)_+\|_{L^\infty(B_1)} < \varepsilon,$$

and then, for any τ such that $\tau \cdot e > \delta$,

$$\partial_\tau u_r(x) > \delta (x \cdot e)_+ - \varepsilon,$$

which combined with the fact that $\partial_\tau u_r \equiv 0$ on $\{u_r = 0\}$, and

$$\Delta(\partial_\tau u_r) = r(\partial_\tau f)(x_0 + r\cdot),$$

allow us to use an *almost positivity property*[12] to deduce that $\partial_\tau u_r \geq 0$ in $B_{1/2}$ for $\delta \gg \varepsilon$ and sufficiently small r.

This monotonicity of u in the τ directions for the cone $\{\tau \cdot e > \delta\}$ implies that the free boundary is a Lipschitz graph in the e direction, with a Lipschitz constant that we can choose as small as we want making $\delta \to 0$.

Thin obstacle problem

The thin obstacle problem arises when we consider the minimization of the Dirichlet energy with a lower dimensional obstacle. It is also called Signorini problem for its connection with the problem of determining the shape of an elastic body resting on a surface. For a nice introduction to the topic, see [91].

The solutions to the minimization problem

$$\min \int_\Omega \frac{|\nabla u|^2}{2}$$

over the set

$$\left\{v \in H^1(\Omega) : v \geq 0 \text{ on } \{x_{n+1} = 0\},\ v|_{\partial\Omega} = g\right\}$$

satisfy the following equation

$$\begin{cases} \Delta u &= 0 \quad \text{in} \quad B_1^+ \\ \min\{u, -\partial_{x_{n+1}} u\} &= 0 \quad \text{on} \quad B_1 \cap \{x_{n+1} = 0\}. \end{cases} \tag{4}$$

Alternatively, we study the problem posed in the whole unit ball $B_1 \subset \mathbb{R}^{n+1}$ (extending by even symmetry) as

$$\begin{cases} \Delta u &= 0 & \text{in} \quad B_1 \setminus \{x_{n+1} = 0\} \\ \min\{u, -\Delta u\} &= 0 & \text{on} \quad B_1 \cap \{x_{n+1} = 0\} \\ u(x', x_{n+1}) &= u(x', -x_{n+1}) & \text{in} \quad B_1, \end{cases}$$

where now Δu needs to be understood in the sense of distributions. Since the obstacle is lower dimensional, we define the free boundary to be the boundary of the contact set in the relative topology of the hyperplane, $\partial\{u|_{\{x_{n+1}=0\}} > \varphi\} \times \{0\}$.

For the Signorini problem, the optimal regularity of solutions is Lipschitz across $\{x_{n+1} = 0\}$, and $C^{1,1/2}$ on each side [7]. As in the *thick* obstacle problem, blow-ups are a key tool to understand the free boundary. However, here the distinction of regular and singular points is not based on the size of the contact set, but rather on the homogeneity of the blow-ups.

Theorem ([91, Theorem 4.4])**.** *Let u be a solution to the thin obstacle problem* (4)*, and assume the origin is a free boundary point. Then,*

$$u_r(x) := \frac{u_r(x_0 + rx)}{\|u\|_{L^2(\partial B_r)}} \to u_0$$

up to subsequences, where u_0 is a global homogeneous solution to the thin obstacle problem with homogeneity degree λ. Moreover, either $\lambda = \frac{3}{2}$ or $\lambda \geq 2$.

[12] An almost positivity property is related to the mean value property. It roughly says that if a harmonic function is positive and big in a big enough subset of B_1, and bounded below by a small negative constant in the whole B_1, then it has to be nonnegative in $B_{1/2}$.

In this setting, when $\lambda = \frac{3}{2}$ we call the point regular, and when $\lambda \geq 2$ we call it degenerate. We also have that the free boundary is locally C^∞ near regular points.

When the thin obstacle problem is posed in the full space, if we restrict to the hyperplane $\{x_{n+1} = 0\}$, u solves an obstacle problem for the nonlocal operator $(-\Delta)^{1/2}$. This connection is particularly useful to use tools from the thin obstacle problem, which is driven by a local differential operator, to study the fractional obstacle problem.

Fully nonlinear obstacle problem

When replacing the Laplacian by a fully nonlinear elliptic operator, the different formulations of the obstacle problem stop being completely equivalent. The following one was studied by Lee in [143].

$$\begin{cases} F(D^2v) & \leq & 0 \\ v & \geq & \varphi \\ F(D^2v) & = & 0 \quad \text{in} \quad \{v > \varphi\}, \end{cases}$$

where F and φ are smooth, $F(D^2\varphi) \leq -c_0 < 0$, and F is uniformly elliptic, which in this setting means

$$\lambda\|N\| \leq F(M+N) - F(M) \leq \Lambda\|N\|,$$

for some $0 < \lambda \leq \Lambda$, and any symmetric matrices M and $N \geq 0$. Then, under these hypotheses, $v \in C^{1,1}$ and the free boundary is $C^{1,\alpha}$ at regular points.

More generally, one can study problems of the form

$$\begin{cases} F(D^2u, x) = f\chi_{\{u>0\}} \\ u \geq 0. \end{cases}$$

This fully nonlinear obstacle problem (and more general ones without the sign condition on u) has been studied by Lee, Shahgholian, Figalli, and more recently by Indrei and Minne in [144, 105, 120]. They proved that if F is convex, f is Lipschitz and $f \geq \tau_0 > 0$, the free boundary is C^1 at regular points.

Moreover, there is also a fully nonlinear version of the Signorini problem [152, 90, 168].

Obstacle problems for nonlocal operators

The obstacle problem can be generalized to integro-differential operators as

$$\min\{Lu, u - \varphi\} = 0 \quad \text{in } \mathbb{R}^n,$$

where L is of the form

$$Lu(x) = \int_{\mathbb{R}^n} \Big(u(x) - u(x+y)\Big)K(y)\mathrm{d}y, \tag{5}$$

where K is even, homogeneous, and satisfies the uniform ellipticity condition

$$\lambda|y|^{-n-2s} \leq K(y) \leq \Lambda|y|^{-n-2s}. \tag{6}$$

In this setting, we have the following dichotomy [46].

Theorem. *Let u be a global solution to $\min\{Lu, u-\varphi\} = 0$, with $\varphi \in C^\infty$. Let x_0 be a point in the free boundary $\partial\{u > \varphi\}$. Then,*

- *Either*
$$0 < cr^{1+s} \leq \|u-\varphi\|_{L^\infty(B_r(x_0))} \leq Cr^{1+s},$$
- *Or*
$$\|u-\varphi\|_{L^\infty(B_r(x_0))} \leq Cr^{1+s+\alpha},$$
for some $\alpha > 0$.

The points that satisfy the first condition are called regular, they form an open subset of the free boundary, and the free boundary is locally a $C^{1,\alpha}$ manifold around them.

Obstacle problems for elliptic integro-differential equations were studied first for the fractional Laplacian, and they were one of the main motivations for the development of the Caffarelli-Silvestre extension [49, 48, 185]. The obstacle problem for $L = (-\Delta)^s$ can be seen as a generalization of the thin obstacle problem, that corresponds to the case $s = \frac{1}{2}$, and moreover one can recover some results in the classical case taking the limit $s \to 1$.

The fractional Laplacian is special in the sense that one can use the extension to convert the nonlocal problem into another local problem, and use techniques from the world of local elliptic equations. To extend the regularity results to a wider class of integro-differential operators, completely new ideas were needed. The foundational result in this area is due to Caffareli, Ros-Oton and Serra, where they extend the known regularity results for the fractional Laplacian to a wide class of integro-differential obstacle problems [46].

Finally, the higher regularity of free boundaries has also been studied, concluding that they are C^∞ near regular points when the obstacle and the kernel of the operator are also smooth [1].

Parabolic obstacle problems

We consider now the version of the parabolic obstacle problem where solutions are nondecreasing in time, that is, the one that arises from the Stefan problem:

$$\begin{cases} u_t - \Delta u &= \chi_{\{u>0\}} \quad \text{in} \quad \Omega \times (0,\infty) \\ u &\geq 0 \\ u_t &\geq 0. \end{cases}$$

The known regularity results mirror the ones for the classical obstacle problem. The optimal regularity of solutions, $C^{1,1}$ in space and C^1 in time, was investigated by Brezis and Kinderlehrer [31], and Caffarelli and Friedman [43]. The regularity of free boundaries was studied in parallel to the one of the elliptic problem by Kinderlehrer, Nirenberg, and Caffarelli [130, 34].

If we drop the assumptions of $u \geq 0$ and $u_t \geq 0$, we have the no-sign parabolic obstacle problem,

$$\partial_t u - \Delta u = f\chi_{\{u \neq 0\}},$$

that has been researched in [44] and [4].

The parabolic Signorini problem was extensively studied by Danielli, Garofalo, Petrosyan and To in [69], where they established the optimal regularity of solutions and studied the free boundary. Further results include higher regularity of free boundaries [14], and obstacles with lower regularity [162].

Parabolic obstacle problems have also been studied for fully nonlinear operators [106, 120, 12].

Nonlocal parabolic obstacle problems

As in the local case, the model parabolic obstacle problem for integro-differential equations has some extra properties coming, in this case, from its motivation by the pricing of American options. We study the problem

$$\begin{cases} \min\{\partial_t u + Lu, u - \varphi\} &= 0 \quad \text{in} \quad \mathbb{R}^n \times (0,T) \\ u(\cdot, 0) &= \varphi \quad \text{in} \quad \mathbb{R}^n, \end{cases}$$

for nonlocal operators of the form (5)-(6).

The character of nonlocal parabolic equations depends crucially on the order of differentiation of the diffusion operator, $2s$. In the subcritical case, $2s > 1$, the highest order operator is the nonlocal one, while in the critical case $2s = 1$ there is a competition between L and ∂_t, and in the supercritical case $2s < 1$ the time derivative has the highest order of differentiation.

These differences of orders of differentiation have a profound impact when performing blow-ups. In the subcritical case, blow-ups produce global solutions of the elliptic problem, the same that happens with local parabolic equations. In the critical case, blow-ups are hyperbolic, which makes them much more delicate, and in the supercritical case there is no known meaningful notion of blow-up.

The regularity of solutions for $L = (-\Delta)^s$ was studied by Caffarelli and Figalli in [42]. Then, the regularity of free boundaries was adressed by Barrios, Figalli and Ros-Oton in the subcritical regime [19]. Finally, Figalli, Ros-Oton and Serra extended the results of [19] to the critical setting in the very recent paper [104].

We deal with the supercritical case in Chapter 1 of this Thesis.

Singular and degenerate points

Obstacle problem

In the obstacle problem, singular points are those where the contact set has zero density. A first (nontrivial) question is whether they exist. The answer is yes, and the first examples were constructed by Schaeffer in [177], where he constructed solutions to the obstacle problem where the free boundary has cusps. The questions were then, how big the singular set could be, and if there were other kinds of singularities.

The first advances were in $\mathbb{R}^2$, where Caffarelli and Rivière proved that the singular set is contained in a C^1 curve [45]. Then, Sakai proved that the only singularities that may appear are isolated cusps that look like the ones constructed by Schaeffer, taking advantage of complex analysis [171, 172].

In higher dimensions, Caffarelli first proved that the blow-up is unique at singular points, and moreover, if x_0 is a singular point,

$$u(x_0 + x) = p_{x_0}(x) + o(|x|^2),$$

where p_{x_0} is a homogeneous quadratic polynomial satisfying $\Delta p_{x_0} = 1$ and $p_{x_0} \geq 0$ [36]. As a consequence, he showed that the singular set is locally contained in a $(n-1)$-dimensional C^1 manifold. This dimensional bound is sharp, since there are examples where the singular set has the same dimension as the full free boundary [177].

Since the blow-up is unique, one can stratify the singular points in terms of the dimension of the linear space $\{p_{x_0}(x) = 0\}$. Then, we can write $\Sigma = \bigcup \Sigma_k$, where Σ is the set of singular free boundary points, and Σ_k are those whose blow-up has a k-dimensional zero set. Then, each of the strata Σ_k are locally contained in a k-dimensional C^1 manifold.

Generic regularity

Since the free boundary has dimension $n-1$ and there are examples where the singular set matches that dimension, the next question is how often can this happen. Schaeffer conjectured that, generically, the free boundary in the obstacle problem is smooth [176].

To prove a generic regularity result, we need to give it a precise mathematical meaning. A convenient approach is to consider one-parameter families of solutions $\{u_\lambda\}$, and to say that a property is generically true if it holds for almost every λ, for all such families.

In the context of free boundary problems, this has been done considering families of solutions that are monotone, with the simplest example being taking $g_\lambda = g + \lambda$ as boundary datum. With this construction, Monneau proved the Schaeffer conjecture in $\mathbb{R}^2$ [154]. This result was extended to $\mathbb{R}^3$ and $\mathbb{R}^4$ by Figalli, Ros-Oton and Serra [102].

Thin obstacle problem

In the thin obstacle problem[13], recall that free boundary points have an associated frequency (homogeneity degree) λ, and are classified into regular points when $\lambda = \frac{3}{2}$, and degenerate points when $\lambda \geq 2$.

The first step towards understanding singular points is classifying global homogeneous solutions in dimension $n+1 = 2$, which can be done explicitly. We find that the solutions have homogeneities belonging to the set

$$\mathcal{S} = \left\{2m - \frac{1}{2}, 2m, 2m+1\right\}, \tag{7}$$

where m are positive integers. When $\lambda = 2m$, solutions are harmonic polynomials, nonnegative on $\{x_2 = 0\}$, and otherwise they are of the form

$$\mathrm{Re}\left((x_1 + i|x_2|)^{2m-\frac{1}{2}}\right), \text{ and } \mathrm{Im}\left((x_1 + i|x_2|)^{2m+1}\right).$$

Points with even frequency are called singular because the contact set has zero density at them, as it happens in the obstacle problem. They were extensively studied by Garofalo and Petrosyan in [110], where they proved that the blow-up is unique, and hence the singular set can be covered by a countable union of $(n-1)$-dimensional C^1 manifolds.

When the homogeneity of a point belongs to the set $\{\frac{7}{2}, \frac{11}{2}, \ldots\}$, it is expected that the free boundary behaves similarly to around regular points, but much less is actually known.

The set of points with odd frequency can be characterized as points where the contact set has density one. They have been studied in [107, 102], among other works, but they are not known to exist as free boundary points: the only known examples are global solutions where the contact set is the full thin space. If they exist, the blow-up is unique and the set can also be covered by a countable union of $(n-1)$-dimensional C^1 manifolds.

[13] In this section we assume the obstacle to be analytic.

Finally, the set of points whose homogeneity does not belong to $\mathcal{S}$ is expected to be empty. The only result in this direction is [65], due to Colombo, Velichkov and Spolaor, where they prove that even frequencies are isolated, that is, for every $m \in \mathbb{Z}^+$, there exists $c_m > 0$ such that there are no points whose frequency is in $(2m - c_m, 2m) \cup (2m, 2m + c_m)$.

Results of the thesis (Part I)

The first part of this Thesis is a collection of results on regularity theory for obstacle problems. Chapter 1 deals with nonlocal parabolic obstacle problems, whereas Chapter 2 is about the thin obstacle problem. Finally, in Chapter 3 we prove estimates for elliptic and parabolic nonlocal equations, with applications in obstacle-type problems.

Optimal regularity for supercritical parabolic obstacle problems

Chapter 1 is devoted to the study of the regularity of solutions and free boundaries in the supercritical parabolic obstacle problem. We consider integro-differential operators of the form (5)-(6), with $s \in (0, \frac{1}{2})$. Recall that $2s < 1$ is what makes the problem supercritical.

Our first main result is the optimal $C^{1,1}$ regularity of solutions:

Theorem. *Let u be a solution to*

$$\begin{cases} \min\{\partial_t u + Lu, u - \varphi\} = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u(\cdot, 0) = \varphi & \text{in } \mathbb{R}^n, \end{cases}$$

where the operator L is of the form (5)-(6), and $\varphi \in C_c^{2,1}(\mathbb{R}^n)$.

Then, u is Lipschitz in $\mathbb{R}^n \times [0,T]$ and

$$u \in C^{1,1}(\mathbb{R}^n \times (0,T]),$$

i.e., the solution u is globally[14] *$C^{1,1}$ in x and t.*

Our result is surprising because the optimal regularity for the parabolic obstacle problem is higher than the optimal $C^{1,s}$ regularity of solutions in the elliptic setting [46]. This happens because, since the initial condition is equal to the obstacle, the solution is always increasing, and u can never be stationary and behave like in the elliptic problem. Nevertheless, as $t \to \infty$, u converges to a solution to the elliptic problem, and hence we cannot expect the $C^{1,1}$ estimates to be uniform in T.

The strategy of the proof is very different not only to the studies on the subcritical and critical cases, [19, 104], but to the majority of existing literature on free boundary problems. Due to the supercritical scaling, blow-ups are ineffective. Instead, our proof is based on the fact that ∂_t is higher order with respect to L, barriers and scaling arguments. Moreover, we do not use any monotonicity formulas, making the techniques applicable to a broad class of operators.

The second main result is the global $C^{1,\alpha}$ regularity of the free boundary as a space-time graph.

Theorem. *In the setting of the previous Theorem,*

- *The free boundary $\partial\{u > \varphi\}$ is a $C^{1,\alpha}$ graph in the t direction, i.e.*

  $$\partial\{u > \varphi\} = \{t = \Gamma(x)\},$$

  with $\Gamma \in C^{1,\alpha}$ and $\alpha > 0$.

[14] Here we mean that for all $t_0 > 0$, $u \in C^{1,1}(\mathbb{R}^n \times [t_0, T])$.

- *If (x_0, t_0) is a free boundary point, the solution admits an expansion*

$$(u - \varphi)(x_0 + x, t_0 + t) = c_0(t - a \cdot x)_+^2 + O(t^{2+\alpha} + |x|^{2+\alpha}),$$

  where $c_0 > 0$, $\alpha > 0$ and $a \in \mathbb{R}^n$.

In the supercritical setting, the regularity of the free boundary is quite a direct consequence of the optimal regularity of the solution. Moreover, an expansion of the same form for *all* free boundary points is a very uncommon result in the context of obstacle problems, where regular and singular points are usually classified based on their different behaviours at leading order.

In contrast, we can define *singular points* as the points (x_0, t_0) such that $a = 0$ in the expansion, meaning that the normal vector to the free boundary points in the time direction, and *regular points* as the rest. As a consequence, we obtain the following generic regularity result.

Theorem. *In the setting of the previous Theorem,*

- *The set of regular free boundary points is an open subset of $\partial\{u > \varphi\}$.*

- *If (x_0, t_0) is a regular free boundary point, the free boundary $\partial\{u > \varphi\}$ is locally a $C^{1,\alpha}$ graph in the x_i direction for some $i \in \{1, \dots, n\}$, i.e.*

$$\partial\{u > \varphi\} \cap B_r(x_0, t_0) = \{x_i = F(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, t)\},$$

  with $F \in C^{1,\alpha}$, $\alpha > 0$ and $r > 0$.

- *Let Σ_t be the set of singular free boundary points (x_0, t_0) with $t_0 = t$. Then,*

$$\mathcal{H}^{n-1}(\Sigma_t) = 0 \quad \text{for almost every} \quad t \in (0, T).$$

The key points in the proofs are the following. Note that the fact that $2s < 1$ is used several times.

- By comparison principle arguments, $D^2 u \geq -C$ and $|\nabla u| \leq C$.

- Since $u \in C^{0,1}$, $u_t = (Lu)^-$. Hence $u_t \in C^{0,\alpha}$ with $\alpha = 1 - 2s > 0$.

- By the elliptic techniques in [46], and heat kernel estimates, knowing that u_t is continuous is enough to prove $u \in C^1$.

- From $u \in C^1$, it follows that u_t is a solution to the Dirichlet problem for the parabolic equation $(\partial_t + L)u_t = 0$ in the set $\{u > \varphi\}$.

- By a continuity argument and an almost positivity property, $|\nabla u| \leq C u_t$ in $\mathbb{R}^n \times [t_1, t_2]$ for all time intervals. It follows that the free boundary is a Lipschitz graph $\{t = \Gamma(x)\}$.

- By barriers in cones, we obtain that u_t grows linearly at the free boundary:

$$0 < c_0(t - t_0) \leq u_t(x_0, t) \leq M(t - t_0),$$

  where (x_0, t_0) is a free boundary point. The fact that $2s < 1$ is crucial here.

- Then, using the fact that $|\nabla u| \leq Cu_t$, if ν is an unit vector close to the time direction,
$$u_\nu((x_0, t_0) + r\nu) \leq 2Mr.$$
- It follows that
$$u_{\nu\nu} \leq C \quad \text{in } \mathbb{R}^n \times [t_1, t_2],$$
by an L^1-L^∞ interior estimate.
- By the semiconvexity of u and linear algebra, $|D^2 u| \leq C$.
- Since $u_t = (Lu)^-$, $u_t \in C^{1,\alpha}$. Here we use supercriticality again.
- Hence, $u_{tt}, u_{ti} \in C^{0,\alpha}$. On the other hand, $u_{tt} \geq c_0 > 0$, and therefore
$$\hat{n} = \frac{(u_{t1}, \ldots, u_{tn}, u_{tt})}{\sqrt{u_{tt}^2 + \sum_{j=1}^n u_{tj}^2}} = \frac{(u_{t1}/u_{tt}, \ldots, u_{tn}/u_{tt}, 1)}{\sqrt{1 + \sum_{j=1}^n u_{tj}^2/u_{tt}^2}} \in C^{0,\alpha},$$
where $\hat{n}$ is the normal vector to the level sets of u_t.
- Since the free boundary coincides with the zero level set of u_t, the free boundary is $C^{1,\alpha}$.

Generic regularity of free boundaries for the thin obstacle problem

In Chapter 2 we study generic regularity properties for the Signorini problem. Our main result establishes that the degenerate set has zero $\mathcal{H}^{n-3-\alpha_0}$ measure for a generic solution. As a consequence, we obtain that, for $n+1 \leq 4$, the free boundary is generically smooth.

The notion of genericity studied here is based on monotone families of solutions. We consider $u : B_1 \times [-1, 1] \to \mathbb{R}$ to be such that $u(\cdot, t)$ is a solution to the thin obstacle problem (4) for each $t \in [-1, 1]$ and

$$\begin{cases} u(\cdot, t') - u(\cdot, t) & \geq & 0 & \text{in} & \overline{B_1} \\ u(\cdot, t') - u(\cdot, t) & \geq & t' - t & \text{on} & \partial B_1 \cap \{|x_{n+1}| \geq \frac{1}{2}\} \\ \|u(\cdot, t)\|_{C^{0,1}(B_1)} & \leq & 1, & & \end{cases} \tag{8}$$

for all $-1 \leq t < t' \leq 1$.

Theorem. *Let $u : B_1 \times [-1, 1] \to \mathbb{R}$ be a solution to (4)-(8). Then, for almost every $t \in [-1, 1]$,*

(a) If $n \leq 3$, $\mathrm{Deg}(u(\cdot, t)) = \emptyset$.

(b) If $n \geq 4$, $\dim_{\mathcal{H}}(\mathrm{Deg}(u(\cdot, t))) \leq n - 3 - \alpha_\circ$, for some $\alpha_\circ > 0$ depending only on n, where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension.

Here, $\mathrm{Deg}(u(\cdot, t))$ denotes the degenerate set for the solution $u(\cdot, t)$. Recall that the free boundary is C^∞ outside of it. As a corollary, we deduce an analogue to a conjecture of Schaeffer for the obstacle problem in $\mathbb{R}^3$ and $\mathbb{R}^4$. This was already known in $\mathbb{R}^2$ [96].

Theorem. *Let $n+1 \leq 4$. Then, generically, the free boundary in the thin obstacle problem (4) is smooth.*

The proof follows the strategy of the generic regularity in the *thick* obstacle problem by Figalli, Ros-Oton and Serra [102], together with a fine analysis of solutions near degenerate points combining some very recent works, [65, 96, 93, 175], and new ideas.

First, we stratify the degenerate set into subsets of different frequencies

$$\mathrm{Deg}(u) = \Gamma_2^{\mathrm{o}}(u) \cup \Gamma_2^{\mathrm{a}}(u) \cup \Gamma_3(u) \cup \Gamma_{\geq 7/2}(u) \cup \Gamma_*(u),$$

where Γ_κ denotes the set of free boundary points with frequency κ, and Γ_* denotes the points whose frequency does not belong to the set $\mathcal{S}$ defined in (7).

In the case of Γ_2, we further divide the set into *ordinary* and *anomalous* quadratic points, depending on the homogeneity of the second leading order in the expansion, as follows.

If x_0 is a free boundary point with frequency 2 and p_2 is the blow-up at that point, then $u(x_0+\cdot) - p_2$ can be approximated at the leading order by a homogeneous harmonic polynomial of degree $\lambda \geq 2$. In this work we call ordinary quadratic points to those with $\lambda \geq 3$, and anomalous quadratic points to those with $\lambda = 2$.

Now, for each of these sets we perform a **dimension reduction argument** to estimate their maximum possible size. The argument is based on an original idea of Federer [88] (in the context of minimal surfaces), and, at a high level, works as follows. Suppose that in dimension k all solutions to a problem with good compactness properties are smooth. Then, in dimension n we can see that the set of singular points has dimension at most $n-k-1$. Indeed, if the dimension of the singular set was higher, there would exist a point where the singular set accumulates in $n-k$ linearly independent directions. Then, by zooming in, we could construct another solution that is independent of those directions and has a singular point, and therefore, could be identified with a solution in dimension k with a singular point, a contradiction. This idea can be adapted to prove that the points do not belong to a particular subset Γ_κ, and, more importantly, this approach is applicable to a monotone family of solutions rather than just a single solution.

The final ingredient is the following geometric measure theory result that we call the **cleaning lemma**.

Lemma ([102, Corollary 7.8]). *Consider the family $\{E_t\}_{t\in[-1,1]}$ with $E_t \subset \mathbb{R}^n$, and let us denote $E := \bigcup_{t\in[-1,1]} E_t$.*

Let $1 \leq \beta \leq n$, and assume that the following holds:

- $\dim_{\mathcal{H}} E \leq \beta$,
- *for all $\varepsilon > 0$, $t_0 \in [-1,1]$, and $x_0 \in E_{t_0}$, there exists $\rho > 0$ such that*

$$B_r(x_0) \cap E_t = \emptyset,$$

  for all $r \in (0,\rho)$ and $t > t_0 + r^{\gamma-\varepsilon}$.

Then,

(a) If $\gamma > \beta$, $\dim_{\mathcal{H}}(\{t : E_t \neq \emptyset\}) \leq \beta/\gamma$.

(b) If $\gamma \leq \beta$, $\dim_{\mathcal{H}}(E_t) \leq \beta - \gamma$, for $\mathcal{H}^1$-a.e. $t \in [-1,1]$.

To apply this lemma, we compute expansions of solutions at free boundary points, and use the monotonicity of the family and the comparison principle to deduce that if x_0 is a free boundary point of $u(\cdot, t_0)$, there exists some $r_0 > 0$ such that u is positive (or identically zero, depending on the case) in one of the following sets:

$$\{x \in B_{r_0} : |x - x_0|^\gamma < t - t_0\} \quad \text{or} \quad \{x \in B_{r_0} : |x - x_0|^\gamma < t_0 - t\},$$

and hence there are no other free boundary points there. The cleaning exponent γ depends on the subset of the free boundary considered.

Finally, applying the cleaning lemma we obtain dimensional bounds on each of the degenerate strata. For $n \geq 4$, the situation can be summarized as follows, where $\alpha, \gamma \in (0,1)$ are dimensional constants, and $\varepsilon > 0$ is an arbitrarily small number.

Set	$\dim_{\mathcal{H}} \cup_t \Gamma$	Cleaning exponent	Generic $\dim_{\mathcal{H}} \Gamma$
Γ_2^{o}	$n-1$	$3-\varepsilon$	$n-4$
Γ_2^{a}	$n-2$	$2-\varepsilon$	$n-4$
Γ_3	$n-1$	$2+\gamma$	$n-3-\gamma$
$\Gamma_{\geq 7/2}$	$n-1$	$5/2-\varepsilon$	$n-7/2$
Γ_*	$n-2$	$1+\alpha$	$n-3-\alpha$

Table 1: Intermediate results obtained for each subset of the free boundary.

For $n = 2$ and $n = 3$, the conclusion is that, generically, the free boundary contains no degenerate points.

Semiconvexity estimates for nonlinear integro-differential equations

Chapter 3 develops a nonlocal analogue to the Bernstein technique to establish semiconvexity estimates for local solutions to general integro-differential equations. Our main applications are elaborating a regularity theory for nonlocal obstacle problems in domains with operators that do not admit an extension, and proving for the first time local semiconvexity estimates for fully nonlinear nonlocal equations. This work answers a question posed by Cabré, Dipierro and Valdinoci in [33]. Finally, we also extend the Bernstein technique to parabolic equations and nonsymmetric operators.

We consider integro-differential operators of the form (5)-(6) with the smoothness conditions

$$|\nabla K(y)| \leq \Lambda |y|^{-1} K(y), \quad |D^2 K(y)| \leq \Lambda |y|^{-2} K(y),$$

and we denote this class of operators by $\mathcal{L}_s(\lambda, \Lambda; 2)$.

The application that originally motivated this project are nonlocal obstacle problems in domains, for which we establish local semiconvexity estimates.

Theorem. *Let $s \in (0,1)$, $L \in \mathcal{L}_s(\lambda, \Lambda; 2)$, and u be any solution to the nonlocal obstacle problem*

$$\min\{Lu, u - \varphi\} = 0 \quad \textit{in } B_1,$$

where $\varphi \in C^4(\mathbb{R}^n)$. Then, u satisfies

$$\partial^2_{ee} u \geq -C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|L\varphi\|_{C^{1,1}(B_1)}) \quad \text{in } B_{1/2}$$

for all $e \in \mathbb{S}^{n-1}$, where C depends only on s, the dimension, and ellipticity constants.

Semiconvexity is an essential tool in the study of nonlocal obstacle problems, because it allows to deduce convexity of blow-ups and to classify them, which is a crucial step in proving optimal regularity of solutions and free boundaries. Up to now, semiconvexity was only known for operators with an extension, or for global solutions. Our result enables then extending the regularity theory for nonlocal obstacle problems to problems in domains for operators that do not admit an extension, and also to study problems where the obstacle has very low regularity, which is new even when the domain is the full space.

Our second main application is the following estimate.

Theorem. *Let $s \in (0,1)$, and let u be any viscosity solution to a fully nonlinear equation*

$$\inf_{\gamma \in \Gamma} \{L_\gamma u\} = 0 \quad \text{in } B_1,$$

where $\{L_\gamma\}_{\gamma\in\Gamma} \subset \mathcal{L}_s(\lambda, \Lambda; 2)$. Then, u satisfies

$$\partial^2_{ee} u \geq -C\|u\|_{L^\infty(\mathbb{R}^n)} \quad \text{in } B_{1/2}$$

for all $e \in \mathbb{S}^{n-1}$, where C depends only on s, the dimension, and ellipticity constants.

The regularity of this kind of nonlocal Bellman equations has been an intriguing open problem since the pioneer works of Caffarelli and Silvestre, where they proved analogues of the Evans-Krylov and Krylov-Safonov theorems, and stated that solutions are $C^{1,\varepsilon} \cap C^{2s+\varepsilon}$ [50, 51, 52]. The higher regularity of solutions is still unknown, even when the operators L_γ have smooth kernels. This semiconvexity result is a one-sided $C^{1,1}$ estimate. It brings attention to the question

Are solutions to nonlocal Bellman equations $C^{1,1}$?

For fully nonlinear second order elliptic equations, semiconvexity implies $C^{1,1}$ regularity. This motivates asking the same question in the nonlocal case.

The Bernstein technique comes from the observation that, if u is a harmonic function, then $|\nabla u|^2$ is subharmonic. This enables the use of the maximum principle to deduce that the value of $|\nabla u|$ in a ball is bounded by its values on the boundary. With a bit more work, one can see that the auxiliary function $w = \eta^2(\partial_e u)^2 + \sigma u^2$ is also subharmonic, for a big enough constant σ that depends on η. If we now choose a cutoff function $\eta \in C^\infty_c(B_1)$ with $\eta \equiv 1$ in $B_{1/2}$, by the maximum principle we get that

$$\|\partial_e u\|^2_{L^\infty(B_{1/2})} \leq \|w\|^2_{L^\infty(B_1)} \leq \|w\|^2_{L^\infty(\partial B_1)} \leq \sigma\|u\|^2_{L^\infty(\partial B_1)},$$

allowing us to estimate the gradient by the L^∞ norm.

In our setting, the key estimates that replace the fact that w is subharmonic in the Bernstein technique are the following.

Theorem. *Let $s \in (s_0, 1)$, with $s_0 > 0$, and $L \in \mathcal{L}_s(\lambda, \Lambda; 2)$. Let $\eta \in C^{1,1}(\mathbb{R}^n)$ be such that $\eta \geq 0$. Then, there exists $\sigma_0 = \sigma_0(n, s_0, \Lambda/\lambda, \|\eta\|_{C^{1,1}(\mathbb{R}^n)}) > 0$ such that for every $\sigma \geq \sigma_0$ and every $u \in C^{1+2s+\varepsilon}_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*

$$L\Big(\eta^2(\partial_e u)^2 + \sigma u^2\Big) \leq 2\eta^2 L(\partial_e u)\partial_e u + 2\sigma L(u)u,$$
$$L\Big(\eta^2(\partial_e u)_+^2 + \sigma u^2\Big) \leq 2\eta^2 L(\partial_e u)(\partial_e u)_+ + 2\sigma L(u)u.$$

To obtain second derivative estimates, we use the estimates replacing u by $\partial_e u$ and adapt the proofs. The importance of the second inequality lies in the application to nonlinear equations, where $L(\partial^2_{ee}u)$ is not necessarily zero but has a sign. This is enough to get semiconvexity estimates.

In the case of second order operators, such inequalities are more or less direct consequences of the product rule. However, in our setting, the result is far from trivial. Our proof starts by **splitting the kernel** of the operator into a singular part with compact support and a bounded part. Taking a cutoff function $\psi \in C^\infty([0, \infty))$ satisfying $0 \leq \psi \leq 1$, $\psi \equiv 0$ in $B_{1/2}$ and $\psi \equiv 1$ in $\mathbb{R}^n \setminus B_1$, we define

$$K_1(y) = \left[1 - \psi\left(\frac{|y|}{\varepsilon}\right)\right] K(y), \quad K_2(y) = \psi\left(\frac{|y|}{\varepsilon}\right) K(y).$$

Then, we define L_1 and L_2 as

$$L_1 u(x) = \int_{\mathbb{R}^n} \Big(u(x) - u(x+y)\Big)K_1(y)\mathrm{d}y, \quad L_2 u(x) = \int_{\mathbb{R}^n} \Big(u(x) - u(x+y)\Big)K_2(y)\mathrm{d}y.$$

It is clear that $L = L_1 + L_2$. Then, we prove the key estimates for L_1 and L_2 separately, and we combine them afterwards. In this step, a crucial observation is that, since the estimates are pointwise, we may choose the splitting threshold ε to depend on the point x.

The inequality for L_2 is proved choosing ε to be comparable to $\eta(x)$, and then by elementary methods. The inequality for L_1 is the part that deals with the differential character of the operator. To prove it, we first rewrite the key estimate

$$L_1\Big(\eta^2(\partial_e u)^2 + \sigma u^2\Big) \leq 2\eta^2 L_1(\partial_e u)\partial_e u + 2\sigma L_1(u)u$$

as

$$L_1(\eta^2)(\partial_e u)^2 - B_1(\eta^2, (\partial_e u)^2) \leq \eta^2 B_1(\partial_e u, \partial_e u) + \sigma B_1(u, u),$$

where B_1 is the bilinear form associated to the operator L_1,

$$B_1(u, v)(x) = \int_{\mathbb{R}^n} \Big(u(x) - u(x+y)\Big)\Big(v(x) - v(x+y)\Big)K_1(y)\,\mathrm{d}y.$$

Now, since B_1 is an integro-differential operator, the highest order of differentiation is on the right-hand side, with the right sign. The difficulty, however, is that in the left-hand side we have a purely pointwise term, $L_1(\eta^2)(\partial_e u)^2$, whereas all the other terms are integrals. To bridge this gap, we use the following **pointwise-to-averaged interpolation inequality**, that can be applied with δ comparable to $\eta(x)$ again.

Lemma. *Let $s \in (0,1)$ and $\delta > 0$. Assume that $K : \mathbb{R}^n \to [0,\infty]$ satisfies for some $0 < \lambda \leq \Lambda$:*

$$\begin{aligned} \lambda|y|^{-n-2s} \leq K(y) \leq \Lambda|y|^{-n-2s} \quad \forall y \in B_\delta, \\ |\nabla K(y)| \leq \Lambda|y|^{-1}K(y) \quad \forall y \in B_\delta. \end{aligned}$$

Then, for every $x \in \mathbb{R}^n$ and $u \in C^{0,1}(B_\delta(x))$ it holds

$$\Big(\partial_e u(x)\Big)^2 \leq \delta^{2s} B_K(\partial_e u, \partial_e u)(x) + c\delta^{2s-2} B_K(u,u)(x),$$

where B_K is the bilinear form as above with kernel K, and $c > 0$ depends only on s, the dimension, and ellipticity constants, but not on δ.

Chapter 1

Optimal regularity for supercritical parabolic obstacle problems

We study the obstacle problem for parabolic operators of the type $\partial_t + L$, where L is an elliptic integro-differential operator of order $2s$, such as $(-\Delta)^s$, in the supercritical regime $s \in (0, \frac{1}{2})$. The best result in this context was due to Caffarelli and Figalli, who established the $C^{1,s}_x$ regularity of solutions for the case $L = (-\Delta)^s$, the same regularity as in the elliptic setting.

Here we prove for the first time that solutions are actually *more* regular than in the elliptic case. More precisely, we show that they are $C^{1,1}$ in space and time, and that this is optimal. We also deduce the $C^{1,\alpha}$ regularity of the free boundary. Moreover, at all free boundary points (x_0, t_0), we establish the following expansion:

$$(u - \varphi)(x_0 + x, t_0 + t) = c_0(t - a \cdot x)_+^2 + O(t^{2+\alpha} + |x|^{2+\alpha}),$$

with $c_0 > 0$, $\alpha > 0$ and $a \in \mathbb{R}^n$.

1.1 Introduction

The aim of this paper is to study the parabolic obstacle problem

$$\begin{cases} \min\{\partial_t u + Lu, u - \varphi\} = 0 & \text{in } \mathbb{R}^n \times (0, T) \\ u(\cdot, 0) = \varphi & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

for nonlocal operators of the form

$$Lu(x) = \int_{\mathbb{R}^n} \big(u(x) - u(x + y)\big) K(y) \mathrm{d}y. \tag{1.2}$$

The kernel K is even and satisfies the uniform ellipticity condition

$$\lambda |y|^{-n-2s} \le K(y) \le \Lambda |y|^{-n-2s}, \quad K(y) = K(-y), \tag{1.3}$$

for some $0 < \lambda \le \Lambda$ and $s \in (0, 1)$. We define the contact set $\{u = \varphi\}$ and the free boundary $\partial\{u > \varphi\}$.

We are mostly interested on studying the *supercritical* case, $s \in (0, \frac{1}{2})$, in which the higher order term is the time derivative instead of the diffusion term. This will give rise to a somewhat unusual approach to the problem, as well as some surprising results.

Nonlocal operators arise naturally when one considers jump-diffusion processes. One of the most classical motivations is the modelling of stock prices, because the nonlocality takes into account the possible large fluctuations of the market. In the trading of options on financial markets, the valuation of American options is an optimal stopping problem. Thus, when the underlying asset price follows a jump-diffusion process, we are led naturally to the parabolic obstacle problem (1.1); see [66, 42] for details. These models were first introduced in the 1970s by Nobel prize winner R. Merton [151], and have been used for many years [178, 66, 157].

1.1.1 The elliptic case

From the mathematical point of view, elliptic and parabolic equations involving jump-diffusion operators have been an active and successful field of research in the past two decades, coming from PDE and from Probability.

The first nonlocal operator of this type to be studied was the fractional Laplacian,

$$(-\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(x+y)}{|y|^{n+2s}} \mathrm{d}y,$$

and problems involving it can be treated as lower-dimensional problems for local operators via the Caffarelli-Silvestre extension[1] [49].

The elliptic obstacle problem,

$$\min\{Lu, u - \varphi\} = 0 \quad \text{in} \quad \Omega,$$

was studied for the case of $L = (-\Delta)^s$ by Caffarelli, Salsa and Silvestre using the extension and local arguments in [48]. Using a new Almgren-type monotonicity formula, they established the optimal $C^{1,s}$ regularity of solutions. Furthermore, they proved the following dichotomy at the free boundary points:

- Either x_0 is a *regular* free boundary point, and

$$cr^{1+s} \leq \sup_{B_r(x_0)} (u - \varphi) \leq Cr^{1+s} \quad \forall r \in (0, r_0),$$

  where $c > 0$.

- Or, if x_0 is not regular, it is called *singular* and then

$$0 \leq \sup_{B_r(x_0)} (u - \varphi) \leq Cr^2 \quad \forall r \in (0, r_0).$$

Moreover, they also proved that the regular points are an open subset of the free boundary and that they are locally a $C^{1,\alpha}$ manifold.

[1] Actually, the paper [49] was motivated by the study of the fractional obstacle problem in [48, 185].

It is important to notice that, in contrast with the classical case $s = 1$, there is no nondegeneracy property of the solutions, i.e. at singular points we may have $\sup_{B_r(x_0)} (u - \varphi) \asymp r^k$ with $k \gg 1$.

The regularity of the free boundary and related questions have been widely investigated in the recent years by several authors. See [110, 20, 107, 65, 93, 175] for more information on the singular points, [133, 72, 123, 134] for higher regularity of the free boundaries, [46, 1] for more general elliptic operators and [159, 111, 95, 139] for operators with drift.

1.1.2 The parabolic case

Much less is known about the parabolic case (1.1). Notice that the problem now depends strongly on the value of s: in the subcritical case $s \in (\frac{1}{2}, 1)$, the higher order term is the nonlocal operator, in the critical case $s = \frac{1}{2}$, both ∂_t and L are of order one, and in the supercritical case $s \in (0, \frac{1}{2})$, the higher order term is the time derivative.

The first result in this direction was the regularity of the solutions in the case $L = (-\Delta)^s$ due to Caffarelli and Figalli [42], where they established the $C^{1,s}$ regularity in x for all $s \in (0, 1)$, and conjectured it to be optimal. They also established the $C^{1,\beta}$ regularity in t, with $\beta = \frac{1-s}{2s} - 0^+$ when $s \geq 1/3$, and that u_t is log-Lipschitz in t when $s < 1/3$. Their proof uses crucially the extension problem for the fractional Laplacian and the $C^{1,s}_x$ regularity is established by using a new monotonicity formula for such problem.

Then, the regularity of the free boundary near regular points was established in the subcritical case, $s \in (\frac{1}{2}, 1)$, by Barrios, Figalli and the first author in [19], where they establish a dichotomy for the free boundary points completely analogous to the elliptic case (in particular, $C^{1,s}_x$ regularity is optimal). One the main difficulties in [19] was to establish a classification of blow-ups in a context where Almgren-type monotonicity formulas are not available.

More recently, Borrin and Marcon established the quasi-optimal regularity of solutions for the subcritical case, $s \in (\frac{1}{2}, 1)$, for a more general equation allowing lower order terms [29].

Despite these developments, in the supercritical case $s \in (0, \frac{1}{2})$ the only known result was the regularity of the solutions for the fractional Laplacian proved in [42]. Quite surprisingly, we prove here that this was not optimal, and that solutions are $C^{1,1}$ in x and t.

1.1.3 Main results

Our main results are the following. We first establish the optimal regularity of the solutions.

Theorem 1.1.1. *Let $n \geq 2$ and $s \in (0, \frac{1}{2})$, and let u be the solution of (1.1) with L an operator satisfying (1.2) and (1.3), and $\varphi \in C^{2,1}_c(\mathbb{R}^n)$.*

Then, u is Lipschitz in $\mathbb{R}^n \times [0, T]$ and

$$u \in C^{1,1}(\mathbb{R}^n \times (0, T]),$$

i.e., the solution u is globally[2] $C^{1,1}$ in x and t.

It is important to notice that because of the initial condition in (1.1), the solution u can never be a solution of the elliptic problem; this is why solutions might be more regular than in

[2]Here we mean that for all $t_0 > 0$, $u \in C^{1,1}(\mathbb{R}^n \times [t_0, T])$.

the elliptic case. Notice also, though, that our solution u to (1.1) always converges as $T \to \infty$ to a solution to the elliptic problem. For this reason, we cannot expect to get a uniform $C^{1,1}$ bound in $\mathbb{R}^n \times (0, \infty)$.

Our proof is completely different from [42], and actually it is mainly based on barriers, comparison principles, and the supercritical scaling of the equation. In particular, we do not use any monotonicity formula, and this allows us not only to get the optimal $C^{1,1}$ regularity for the fractional Laplacian but also to extend the result to general integro-differential operators.

Then, we prove the global $C^{1,\alpha}$ regularity of the free boundary.

Theorem 1.1.2. *Let $n \geq 2$ and $s \in (0, \frac{1}{2})$, and let u be the solution of (1.1) with L an operator satisfying (1.2) and (1.3), and $\varphi \in C_c^{2,1}(\mathbb{R}^n)$. Then,*

- *The free boundary $\partial\{u > \varphi\}$ is a $C^{1,\alpha}$ graph in the t direction,*

$$\partial\{u > \varphi\} = \{t = \Gamma(x)\}$$

  with $\Gamma \in C^{1,\alpha}$ and $\alpha > 0$.

- *If (x_0, t_0) is any free boundary point, the solution admits an expansion*

$$(u - \varphi)(x_0 + x, t_0 + t) = c_0(t - a \cdot x)_+^2 + O(t^{2+\alpha} + |x|^{2+\alpha}), \tag{1.4}$$

  where $c_0 > 0$, $\alpha > 0$ and $a \in \mathbb{R}^n$.

To have that *all* free boundary points have the same expansion is a very uncommon result in the context of obstacle problems, and it contrasts notably with the elliptic and the parabolic subcritical obstacle problems. Moreover, the blow-up techniques that are always used to study free boundaries appeared ineffective here, and our proof of Theorem 1.1.2 uses Theorem 1.1.1 and the fact that L has order $2s < 1$ to gain further regularity instead.

This global regularity result allows us to define regular and singular points *a posteriori* in a very simple way: we say that a free boundary point (x_0, t_0) is regular if the vector a in the expansion (1.4) is not zero, and is singular if $a = 0$.

Finally, as a consequence of Theorem 1.1.2, we deduce that the free boundary is $C^{1,\alpha}$ in the x direction near regular points, and that singular points are in some sense scarce.

Theorem 1.1.3. *Let $n \geq 2$ and $s \in (0, \frac{1}{2})$, and let u be the solution of (1.1) with L an operator satisfying (1.2) and (1.3), and $\varphi \in C_c^{2,1}(\mathbb{R}^n)$. Then,*

- *The set of regular free boundary points is an open subset of $\partial\{u > \varphi\}$.*

- *If (x_0, t_0) is a regular free boundary point, the free boundary $\partial\{u > \varphi\}$ is locally a $C^{1,\alpha}$ graph in the x_i direction for some $i \in \{1, \ldots, n\}$,*

$$\partial\{u > \varphi\} \cap B_r(x_0, t_0) = \{x_i = F(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, t)\},$$

  with $F \in C^{1,\alpha}$, $\alpha > 0$ and $r > 0$.

- *Let Σ_t be the set of singular free boundary points (x_0, t_0) with $t_0 = t$. Then,*

$$\mathcal{H}^{n-1}(\Sigma_t) = 0 \quad \textit{for almost every} \quad t \in (0, T).$$

This problem is very different than the rest of elliptic and parabolic free boundary problems. Notice how Theorem 1.1.2 establishes a regularity result common to regular and singular free boundary points, which deeply contrasts with how these problems were approached until now. Besides, the fact that the free boundary is globally a $C^{1,\alpha}$ graph in the t direction could also be true in the subcritical ($s > 1/2$) case, but is not known in the latter setting.

Remark 1.1.4. There is more literature available for the related (but not equivalent) obstacle problem with operator $(\partial_t - \Delta)^s$. It appears when one considers the parabolic thin obstacle problem ($s = \frac{1}{2}$) or the parabolic thin obstacle problem with a weight. In this setting, the diffusion term is always the highest order term and thus the scaling is always subcritical. For more information on the topic, see [8, 14, 69, 9, 191] and references therein.

1.1.4 Plan of the paper

The paper is organized as follows.

In Section 1.2 we prove a comparison principle and the semiconvexity of solutions. Then, in Section 1.3 we prove that the solutions to (1.1) are C^1, and in Section 1.4, we show that the optimal regularity is $C^{1,1}$. Finally, Section 1.5 is devoted to proving the $C^{1,\alpha}$ regularity of the free boundary and Theorem 1.1.3.

Besides, we include some technical tools in two appendices. Appendix 1.6 includes several regularity and growth estimates for the linear nonlocal parabolic equation, and Appendix 1.7 is a discussion about the penalized obstacle problem.

1.2 Preliminaries and semiconvexity

In this Section we give some basic definitions and prove some basic results that will be used later on.

Given any solution u of (1.1), we define

$$v(x,t) = u(x,t) - \varphi(x).$$

Notice that $\partial_t u = \partial_t v$. Let $B_r(x_0)$ be the ball of radius r and center x_0 in $\mathbb{R}^n$, and let $Q_r(x_0,t_0)$ be the following parabolic cylinders:

$$Q_r(x_0,t_0) = B_r(x_0) \times (t_0 - r^{2s}, t_0 + r^{2s})$$

When the balls or cylinders are centered at the origin we will just write $B_r := B_r(0)$ and $Q_r := Q_r(0,0)$.

We will denote $\nabla := \nabla_x$, and we will write $\nabla_{x,t}$ when we refer to the gradient in all variables.

We will also define the following weighted L^1 norm:

$$\|u\|_{L^1_s} = \|u\|_{L^1_s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}} \mathrm{d}x$$

and the corresponding weighted Lebesgue space

$$L^1_s(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{R}, f \text{ measurable}, \|f\|_{L^1_s} < +\infty\}.$$

Throughout the paper we will assume $n \geq 2$.

1.2.1 Basic tools

We recall some standard tools for elliptic and parabolic PDE that are useful to deal with problem (1.1). Let us start with the comparison principle.

Theorem 1.2.1. *Let L be a nonlocal operator satisfying (1.2) and (1.3), let φ and ψ be uniformly Lipschitz and bounded, and let u and v be the solutions of the following parabolic problems:*

$$\begin{cases} \min\{\partial_t u + Lu, u - \varphi\} &= 0 \quad \text{in} \quad \mathbb{R}^n \times (0,T) \\ u(\cdot,0) &= \varphi \quad \text{in} \quad \mathbb{R}^n, \end{cases}$$

$$\begin{cases} \min\{\partial_t v + Lv, v - \psi\} &= 0 \quad \text{in} \quad \mathbb{R}^n \times (0,T) \\ v(\cdot,0) &= \psi \quad \text{in} \quad \mathbb{R}^n. \end{cases}$$

Assume additionally that $\varphi \le \psi$. Then, $u \le v$ in $\mathbb{R}^n \times (0,T)$.

To prove it, we use the penalization method. This approximation technique is based in considering the solutions to the obstacle problem as the limit of the solutions to the following parabolic problem

$$\begin{cases} \partial_t u^\varepsilon + Lu^\varepsilon &= \beta_\varepsilon(u^\varepsilon - \varphi) \quad \text{in} \quad \mathbb{R}^n \times (0,T) \\ u^\varepsilon(\cdot,0) &= \varphi + \sqrt{\varepsilon}, \end{cases} \tag{1.5}$$

where $\beta_\varepsilon(z) = e^{-z/\varepsilon}$.

Lemma 1.2.2. *Let L be an operator satisfying (1.2) and (1.3), let $\varphi \in C_c^{2,1}(\mathbb{R}^n)$ and let u^ε be the solution of (1.5).*

Then, $u^\varepsilon \to u^0$ as $\varepsilon \to 0$ locally uniformly, where u^0 is the solution of (1.1).

We give the proof in Appendix 1.7. Using this technique, we can now proceed.

Proof of Theorem 1.2.1. It suffices to write u and v as the limits of the penalized versions of the respective problems, and then apply Lemma 1.7.1. □

The following observation is based in the strong maximum principle and will be important in our discussion.

Lemma 1.2.3. *Let u be a solution of (1.1) with L an operator satisfying (1.2) and (1.3), and $\varphi \in C_c^{0,1}(\mathbb{R}^n)$. Then,*

$$u_t > 0 \quad \text{in} \quad \{u > \varphi\}.$$

Proof. First, we see that u is nondecreasing in t. Consider the function $\tilde{u}(x,t) = u(x,t+\delta)$, $\delta > 0$. Then, $\tilde{u}$ is clearly also a solution of $\min\{(\partial_t + L)\tilde{u}, \tilde{u} - \varphi\} = 0$, and $\tilde{u}(\cdot,0) = u(\cdot,\delta) \ge u(\cdot,0) = \varphi$. Hence, $\tilde{u}$ is a supersolution of (1.1), and thus $\tilde{u} \ge u$. This yields $u(x,t+\delta) \ge u(x,t)$ for all x, t and $\delta > 0$.

Let $w = u_t$. Differentiating (1.1), we have

$$\partial_t w + Lw = 0 \quad \text{in} \quad \{u > \varphi\}.$$

We also know that $w \ge 0$ because u is nondecreasing in time. Suppose $w = 0$ at $(x,t) \in \{u > \varphi\}$. Then, by the strong maximum principle, $w \equiv 0$ in all the connected component of (x,t). In particular, $w = 0$ in the segment $\{x\} \times [0,t]$ because each point in the segment belongs either to the contact set or to the connected component of (x,t) in $\{u > \varphi\}$. Hence, $u(x,t) = u(x,0) = \varphi(x)$, contradicting $(x,t) \in \{u > \varphi\}$. Therefore, $w > 0$ in $\{u > \varphi\}$. □

1.2.2 Semiconvexity

An essential property of the solutions is that they are semiconvex, see [19, Lemma 2.1] for the case $L = (-\Delta)^s$ with $s > \frac{1}{2}$. Here we can use the same strategy to prove it.

Proposition 1.2.4. *Let $s \in (0, \frac{1}{2})$, and let u be a solution of (1.1), with L an operator satisfying (1.2) and (1.3), and $\varphi \in C_c^{2,1}(\mathbb{R}^n)$. Then, u is semiconvex, i.e., for all unit vectors e in x, t, $\partial_{ee} u \geq -\hat{C}$, with a uniform bound that depends only on φ, n, s and the ellipticity constants.*

Remark 1.2.5. The assumption $s \in (0, \frac{1}{2})$ can be substituted by the more general $s \in (0, 1)$ and $\varphi \in C_c^{\max\{2, 4s+\varepsilon\}}$ for some small $\varepsilon > 0$.

Proof of Proposition 1.2.4. Using Lemma 1.2.2, we can write u as the limit of solutions to the penalized problem (1.5). Since the locally uniform limit of uniformly semiconvex functions is semiconvex, we only need to prove it for the approximations u^ε.

First, we use Lemma 1.7.5 and notice that $\beta_\varepsilon'' \geq 0$ to obtain

$$\partial_t u^\varepsilon_{\nu\nu} + L u^\varepsilon_{\nu\nu} \geq \beta_\varepsilon'(u^\varepsilon - \varphi)(u^\varepsilon_{\nu\nu} - \varphi_{\nu\nu}),$$

for any unit vector $\nu \in \mathbb{R}^n \times \mathbb{R}$, and also

$$\begin{aligned} u_t^\varepsilon(\cdot, 0) &= e^{-1/\sqrt{\varepsilon}} - L\varphi, \\ u_{tt}^\varepsilon(\cdot, 0) &= L^2\varphi - \frac{1}{\varepsilon} e^{-1/\sqrt{\varepsilon}}(e^{-1/\sqrt{\varepsilon}} - L\varphi). \end{aligned}$$

Define $C_0 := \|u^\varepsilon_{\nu\nu}(\cdot, 0)\|_{L^\infty(\mathbb{R}^n)}$. Then,

$$\begin{aligned} C_0 &\leq \|D_x^2 u^\varepsilon(\cdot, 0)\|_{L^\infty(\mathbb{R}^n)} + \|\nabla u_t^\varepsilon(\cdot, 0)\|_{L^\infty(\mathbb{R}^n)} + \|u_{tt}^\varepsilon(\cdot, 0)\|_{L^\infty(\mathbb{R}^n)} \\ &\leq \|D^2\varphi\|_{L^\infty(\mathbb{R}^n)} + \|\nabla L\varphi\|_{L^\infty(\mathbb{R}^n)} + \|L^2\varphi - \frac{1}{\varepsilon} e^{-1/\sqrt{\varepsilon}}(e^{-1/\sqrt{\varepsilon}} - L\varphi)\|_{L^\infty(\mathbb{R}^n)} \\ &\leq \|D^2\varphi\|_{L^\infty(\mathbb{R}^n)} + \|\nabla L\varphi\|_{L^\infty(\mathbb{R}^n)} + \|L^2\varphi\|_{L^\infty(\mathbb{R}^n)} + C\varepsilon + \|L\varphi\|_{L^\infty(\mathbb{R}^n)} \\ &\leq C\|\varphi\|_{C^{1,1}(\mathbb{R}^n)} + C\varepsilon. \end{aligned}$$

Using again that $\beta_\varepsilon' \leq 0$, it follows that $\beta_\varepsilon'(u^\varepsilon - \varphi)(u^\varepsilon_{\nu\nu} + C_0) \geq 0$ whenever $u^\varepsilon_{\nu\nu} + C_0 \leq 0$. Hence, $w := \min\{0, u^\varepsilon_{\nu\nu} + C_0\}$ satisfies

$$\partial_t w + Lw \geq 0 \quad \text{in} \quad \mathbb{R}^n \times (0, T).$$

Finally, $w \equiv 0$ at $t = 0$ by construction, hence, by the maximum principle, $w \equiv 0$ everywhere, i.e. $u^\varepsilon_{\nu\nu} \geq -C_0$. Since this constant does not depend on ε, we can pass to the limit to get the desired result. □

1.3 C^1 regularity of solutions

Here we prove that solutions u to the problem (1.1) are globally C^1 in x and t. This was already known in the case of $L = (-\Delta)^s$ thanks to [42]; here we prove it in a different way for our general class of operators (1.2). The first step is to prove global Lipschitz regularity.

Notice that we already know that u is Lipschitz because it is globally bounded and semiconvex, but we provide a simple proof to obtain the optimal Lipschitz constant under the minimal requirements for φ.

Proposition 1.3.1. *Let $s \in (0, \frac{1}{2})$, and let u be a viscosity solution of (1.1) with L an operator satisfying (1.2) and (1.3), and $\varphi \in C_c^{0,1}(\mathbb{R}^n)$. Then, u is globally Lipschitz,*

$$\|\nabla u\|_{L^\infty(\mathbb{R}^n\times(0,T))} \le \|\varphi\|_{C^{0,1}(\mathbb{R}^n)} \quad \text{and} \quad \|u_t\|_{L^\infty(\mathbb{R}^n\times(0,T))} \le C\|\varphi\|_{C^{0,1}(\mathbb{R}^n)},$$

where C depends only on the dimension, s and the ellipticity constants.

Proof. First of all, $\|u\|_{L^\infty(\mathbb{R}^n\times(0,T))} \le \|\varphi\|_{L^\infty(\mathbb{R}^n\times(0,T))}$ by Theorem 1.2.1.

We will treat Lipschitz regularity in x and t separately. For spatial regularity, observe that for every $h \in \mathbb{R}^n$, the function $w_h(x,t) := u(x+h,t) + \|\varphi\|_{C^{0,1}}|h|$ is a solution of

$$\begin{cases} \min\{\partial_t w_h + Lw_h, w_h - \varphi_h\} &= 0 \quad \text{in} \quad \mathbb{R}^n \times (0,T] \\ w_h(\cdot,0) &= \varphi_h \quad \text{in} \quad \mathbb{R}^n, \end{cases}$$

with $\varphi_h(x) = \varphi(x+h) + \|\varphi\|_{C^{0,1}}|h| \ge \varphi$. Then, by Theorem 1.2.1, $u \le w_h$ for all h, and it follows that

$$u(x,t) \le u(x+h,t) + \|\varphi\|_{C^{0,1}}|h| \quad \Rightarrow \quad \frac{u(x,t) - u(x+h,t)}{|h|} \le \|\varphi\|_{C^{0,1}}.$$

Since x and h are arbitrary, the Lipschitz regularity follows.

On the other hand, concerning u_t, it is zero in the interior of the contact set, and outside of it $u_t = -Lu$. Moreover, since u is continuous, the contact set is closed and we can estimate the Lipschitz character of u in the t direction knowing it outside of the contact set. Hence, $\|u_t\|_{L^\infty(\mathbb{R}^n\times(0,T))} \le \|Lu\|_{L^\infty(\mathbb{R}^n\times(0,T))}$. Then, we can compute Lu. We omit the time dependence to unclutter the notation.

$$\begin{aligned} |Lu(x)| &= \left| \int_{\mathbb{R}^n} (u(x) - u(x+y))K(y)\mathrm{d}y \right| \\ &\le \left| \int_{B_1} (u(x) - u(x+y))K(y)\mathrm{d}y \right| + \left| \int_{B_1^c} (u(x) - u(x+y))K(y)\mathrm{d}y \right| \\ &\le \int_{B_1} \|\nabla u\|_{L^\infty(\mathbb{R}^n\times(0,T))}|y|K(y)\mathrm{d}y + \int_{B_1^c} 2\|u\|_{L^\infty(\mathbb{R}^n\times(0,T))}K(y)\mathrm{d}y \\ &\le C_1\|\nabla u\|_{L^\infty(\mathbb{R}^n\times(0,T))} + C_2\|u\|_{L^\infty(\mathbb{R}^n\times(0,T))} \le C\|\varphi\|_{C^{0,1}(\mathbb{R}^n)}. \end{aligned}$$

Here we used that $K(y) \le \Lambda|y|^{-n-2s}$ and $s < \frac{1}{2}$, so that $K(y)$ is integrable at infinity and $|y|K(y)$ is integrable near the origin, and finally we applied the previous estimates for $\|\nabla u\|_{L^\infty}$ and $\|u\|_{L^\infty}$ in terms of $\|\varphi\|_{C^{0,1}}$. □

Then, we improve the regularity up until $C^{1,\alpha}$ in t and C^1 in x. We start with the time regularity.

Proposition 1.3.2. *Let $s \in (0, \frac{1}{2})$, and let u be the solution of (1.1) with L an operator satisfying (1.2) and (1.3), and $\varphi \in C_c^{0,1}(\mathbb{R}^n)$. Then, $u_t \in C^\alpha$ and*

$$[u_t]_{C^\alpha(\mathbb{R}^n\times(0,T))} \le C\|\varphi\|_{C^{0,1}(\mathbb{R}^n)},$$

where $\alpha = 1 - 2s > 0$ and C depends only on the dimension, s and the ellipticity constants. Moreover, we have

$$u_t = (Lu)^- \quad \text{in} \quad \mathbb{R}^n \times (0,T).$$

Proof. Let us prove the following estimates for Lu to begin. We prove the spatial regularity first, omitting the time dependence for simplicity of reading.

$$\begin{aligned}
|Lu(x_1) - Lu(x_2)| &= \left| \int_{\mathbb{R}^n} (u(x_1) - u(x_2) - u(x_1+y) + u(x_2+y))K(y)\mathrm{d}y \right| \\
&\le \int_{B_r} \left(|u(x_1) - u(x_1+y)| + |u(x_2) - u(x_2+y)|\right) K(y)\mathrm{d}y \\
&+ \int_{B_r^c} \left(|u(x_1) - u(x_2)| + |u(x_1+y) - u(x_2+y)|\right) K(y)\mathrm{d}y \\
&\le \left(\int_{B_r} 2|y|K(y)\mathrm{d}y + \int_{B_r^c} 2|x_1 - x_2|K(y)\mathrm{d}y \right) \|\nabla u\|_{L^\infty(\mathbb{R}^n\times(0,T))} \\
&\le C(r^{1-2s} + |x_1 - x_2| r^{-2s}) \|\nabla u\|_{L^\infty(\mathbb{R}^n\times(0,T))} \\
&\le C\|\varphi\|_{C^{0,1}(\mathbb{R}^n\times(0,T))} |x_1 - x_2|^{1-2s}.
\end{aligned}$$

In the last steps we used that $|K(y)| \le \Lambda|y|^{-n-2s}$, with $s \in (0, \frac{1}{2})$, we chose $r = |x_1 - x_2|$ and we used the estimate from Proposition 1.3.1.

Then, we prove temporal regularity:

$$\begin{aligned}
&|Lu(x,t_1) - Lu(x,t_2)| \\
&= \left| \int_{\mathbb{R}^n} (u(x,t_1) - u(x,t_2) - u(x+y,t_1) + u(x+y,t_2))K(y)\mathrm{d}y \right| \\
&\le \int_{B_r} \left(|u(x,t_1) - u(x+y,t_1)| + |u(x,t_2) - u(x+y,t_2)|\right) K(y)\mathrm{d}y \\
&+ \int_{B_r^c} \left(|u(x,t_1) - u(x,t_2)| + |u(x+y,t_1) - u(x+y,t_2)|\right) K(y)\mathrm{d}y \\
&\le \int_{B_r} 2|y|K(y)\|\nabla u\|_{L^\infty(\mathbb{R}^n\times(0,T))}\mathrm{d}y + \int_{B_r^c} 2|t_1 - t_2|K(y)\|u_t\|_{L^\infty(\mathbb{R}^n\times(0,T))}\mathrm{d}y \\
&\le C\|\varphi\|_{C^{0,1}(\mathbb{R}^n\times(0,T))} |t_1 - t_2|^{1-2s}.
\end{aligned}$$

Here $r = |t_1 - t_2|$ and the rest of the estimates are used analogously.

Hence, $[Lu]_{C^\alpha(\mathbb{R}^n\times(0,T))} \le C\|\varphi\|_{C^{0,1}(\mathbb{R}^n)}$. In particular, Lu is continuous. Then, recall that $u_t + Lu = 0$ in the set $\{u > \varphi\}$. Moreover, by Lemma 1.2.3, $u_t > 0$ in this set, and therefore $Lu < 0$.

In the interior of the contact set, however, $u(x,t) \equiv \varphi(x)$ and $u_t \equiv 0$. Moreover, $u_t + Lu \ge 0$, and it follows that $Lu \ge 0$ in the interior of the contact set.

By continuity of Lu, $Lu = 0$ on the free boundary. Then, $u_t = 0$ on the free boundary as well.

We deduce that

$$u_t = (Lu)^-$$

and thus $[u_t]_{C^\alpha(\mathbb{R}^n\times(0,T))} \le [Lu]_{C^\alpha(\mathbb{R}^n\times(0,T))}$, as wanted. □

Then, we continue with the regularity in x. First, we need the following estimate, analogous to the elliptic estimate [46, Lemma 2.3].

Lemma 1.3.3. *Let $s \in (0,1)$. There exist constants $\tau \in (0,s)$ and $\delta > 0$ such that the following holds.*

Let v be a globally Lipschitz solution of

$$\begin{cases} v & \geq & 0 & \textit{in} & \mathbb{R}^n \times (-1,0] \\ \partial_{\nu\nu} v & \geq & -\delta & \textit{in} & Q_2 \cap \{t \leq 0\}, \quad \textit{for all} \quad \nu \in \mathbb{S}^{n-1} \\ (\partial_t + L)(v - T_h v) & \leq & \delta|h| & \textit{in} & \{v > 0\} \cap Q_2 \cap \{t \leq 0\}, \end{cases}$$

where T_h is the translation operator defined for any h in the x directions, i.e. $T_h v(x,t) = v(x+h,t)$, and L is a nonlocal operator satisfying (1.2) and (1.3).

Assume that $v(0,t) = 0$ for $t \leq 0$ and that $\sup_{Q_R} |\nabla v| \leq R^\tau$ for all $R \geq 1$. Then,

$$\sup_{B_r \times (-r^{2s},0]} |\nabla v| \leq 2r^\tau,$$

for all $r > 0$. The constants τ and δ depend only on the dimension, s and the ellipticity constants.

Proof. Let $W_r = B_r \times (-r^{2s}, 0]$ be the *past cyilinders at the origin.*

We define

$$\theta(r) := \sup_{r' \geq r} (r')^{-\tau} \sup_{W_{r'}} |\nabla v|.$$

Notice that $\theta(r) \leq 1$ for $r \geq 1$ because $\sup_{W_R} |\nabla v| \leq \sup_{Q_R} |\nabla v| \leq R^\tau$ for $R \geq 1$. The result we aim to prove is equivalent to showing $\theta(r) \leq 2$ for all $r \in (0,1)$. Observe also that θ is nonincreasing by definition.

Assume by contradiction that $\theta(r) > 2$ for some r. Then, by construction there exists $r_0 \in (r,1)$ such that

$$\theta(r_0) \geq r_0^{-\tau} \sup_{W_{r_0}} |\nabla v| \geq (1-\varepsilon)\theta(r) \geq (1-\varepsilon)\theta(r_0) \geq \frac{3}{2},$$

where $\varepsilon > 0$ is to be chosen later.

Then, we define the scaling

$$v_0(x,t) := \frac{v(r_0 x, r_0^{2s} t)}{\theta(r_0) r_0^{1+\tau}}.$$

Let $\tau \in (0,s)$. Then, the rescaled function satisfies

$$\begin{cases} v_0 & \geq & 0 & & & \text{in} & \mathbb{R}^n \times (-2r_0^{-2s}, 0] \\ \partial_{\nu\nu} v_0 & \geq & -r_0^{2-1-\tau}\delta & \geq & -\delta & \text{in} & Q_{2/r_0} \\ (\partial_t + \tilde{L})(v_0 - T_h v_0) & \leq & r_0^{2s-1-\tau}\delta|r_0 h| & \leq & \delta|h| & \text{in} & \{v_0 > 0\} \cap Q_{2/r_0}, \end{cases}$$

where $\tilde{L}$ is the corresponding nonlocal operator with the appropriate scaled kernel, and it has the same ellipticity constants. Notice that $\|\nabla v_0\|_{L^\infty(W_1)} \leq 1$ by construction.

Moreover, by the definition of θ and r_0, for all $R \geq 1$ the following estimates hold:

$$1 - \varepsilon \leq \sup_{|h| \leq \frac{1}{4}} \sup_{W_1} \frac{v_0(x,t) - v_0(x+h,t)}{|h|} \quad \text{and} \quad \sup_{|h| \leq \frac{1}{4}} \sup_{W_R} \frac{v_0 - T_h v_0}{|h|} \leq (R + \frac{1}{4})^\tau.$$

Let $\eta \in C_c^2(Q_{3/2})$ with $\eta \equiv 1$ in Q_1 and $0 \le \eta \le 1$. Then,

$$\sup_{|h|\le\frac{1}{4}} \sup_{W_1} \left(\frac{v_0 - T_h v_0}{|h|} + 3\varepsilon\eta \right) \ge 1 + 2\varepsilon.$$

Notice that if $\tau > 0$ is small enough,

$$\sup_{|h|\le\frac{1}{4}} \sup_{W_3} \frac{v_0 - T_h v_0}{|h|} \le (3 + \frac{1}{4})^\tau < 1 + \varepsilon.$$

Then, we can choose $h_0 \in B_{1/4}$ such that

$$M := \max_{W_{3/2}} \left(\frac{v_0 - T_{h_0} v_0}{|h_0|} + 3\varepsilon\eta \right) \ge 1 + \varepsilon,$$

and the maximum is attained at a point (x_0, t_0) where $\eta(x_0, t_0) > 0$.

Define

$$w := \frac{v_0 - T_{h_0} v_0}{|h_0|}.$$

By construction, $w + 3\varepsilon\eta \le M$ in $W_{3/2}$ and in $W_3 \setminus W_{3/2}$. Therefore, $w + 3\varepsilon\eta \le M$ in $Q_3 \cap \{t \le 0\}$. Besides, $v_0(x_0, t_0) > 0$ because if not $w(x_0, t_0) < 0$ and then $w + 3\varepsilon\eta < 1 + \varepsilon$.

Now we evaluate the equation at (x_0, t_0) to obtain a contradiction.

On the one hand, since (x_0, t_0) is a maximum of $w + 3\varepsilon\eta$, and (x_0, t_0) is either an interior point of $W_{3/2}$ or a point in $B_{3/2} \times \{0\}$,

$$\partial_t(w + 3\varepsilon\eta) \ge 0.$$

On the other hand, we can use the semiconvexity of v_0, together with $v_0(0, t) = 0$ for $t \le 0$ to obtain a lower bound for $\tilde{L}w$. Let $e = \frac{h_0}{|h_0|}$ and $k = |h_0|$. Then, for $x \in B_1$ and omitting the dependence on t,

$$v_0(x) \le \frac{k v_0(0) + |x| v_0\left(x + k\frac{x}{|x|}\right)}{k + |x|} + \frac{k\delta}{2}|x|^2 \le v_0\left(x + k\frac{x}{|x|}\right) + \delta,$$

using that $|x| < 1$ and $k < 1$. Then, combining this fact with the definition of w,

$$w(x) = \frac{v_0(x) - v_0(x + ke)}{k} \le \frac{v_0\left(x + k\frac{x}{|x|}\right) - v_0(x + ke)}{k} + \delta \le \left| \frac{x}{|x|} + e \right| + \delta,$$

for all $x \in B_1$, where we also used that $\|\nabla v_0\|_{L^\infty(W_1)} \le 1$. In particular, $w(x, t) < \frac{1}{2}$ for all $t \le 0$ when $\delta < \frac{1}{4}$ and

$$x \in C_e := \left\{ x \in B_1 : \left| \frac{x}{|x|} + e \right| < \frac{1}{4} \right\}.$$

Using that $M \ge 1 + \varepsilon$ and $w < 1 + \varepsilon$ in W_3,

$$1 - 2\varepsilon \le w(x_0, t_0) < 1 + \varepsilon.$$

Moreover, $w + 3\varepsilon\eta$ has a maximum at (x_0, t_0) (global in $B_3 \times \{t_0\}$), and hence

$$w(x_0, t_0) - w(x, t_0) \geq -3\varepsilon|D^2\eta(x_0, t_0)|\frac{|x - x_0|^2}{2} = -C\varepsilon|x - x_0|^2,$$

for all $x \in B_3$.

Let us now compute $\tilde{L}w$ at the point (x_0, t_0). Using the previous estimates,

$$\begin{aligned}
\tilde{L}w(x_0, t_0) &= \int_{\mathbb{R}^n} (w(x_0, t_0) - w(x_0 + y, t_0))K(y)\mathrm{d}y \\
&\geq \lambda \int_{\mathbb{R}^n} (w(x_0, t_0) - w(x_0 + y, t_0))_+ |y|^{-n-2s}\mathrm{d}y \\
&\quad - \Lambda \int_{\mathbb{R}^n} (w(x_0, t_0) - w(x_0 + y, t_0))_- |y|^{-n-2s}\mathrm{d}y \\
&\geq \lambda \int_{C_e - x_0} (\frac{1}{2} - 2\varepsilon)|y|^{-n-2s}\mathrm{d}y - \Lambda \int_{B_{3/2}} C\varepsilon|y|^2|y|^{-n-2s}\mathrm{d}y \\
&\quad - \Lambda \int_{B_{3/2}^c} ((|y| + \frac{3}{2})^\tau - 1 + 2\varepsilon)|y|^{-n-2s}\mathrm{d}y \\
&\geq c - C\varepsilon - \Lambda \int_{B_{3/2}^c} ((|y| + \frac{3}{2})^\tau - 1)|y|^{-n-2s}\mathrm{d}y \geq c - C\varepsilon,
\end{aligned}$$

where in the last step we choose $\tau > 0$ even smaller if needed to absorb the integral into the $C\varepsilon$ term.

Finally,

$$(\partial_t + \tilde{L})w(x_0, t_0) \geq -3\varepsilon\eta_t(x_0, t_0) + c - C\varepsilon > \delta,$$

choosing small enough ε and δ, reaching a contradiction. Hence, $\theta(r) \leq 2$ for all $r \in (0, 1)$, as we wanted to prove. □

Now we can apply Lemma 1.3.3 to obtain C^1 regularity.

Proposition 1.3.4. *Let $s \in (0, \frac{1}{2})$, and let u be the solution of (1.1) with L an operator satisfying (1.2) and (1.3), and $\varphi \in C_c^{2,1}(\mathbb{R}^n)$. Then, $\nabla u \in C(\mathbb{R}^n \times (0, T))$. In particular, $u \in C^1(\mathbb{R}^n \times (0, T))$.*

Proof. First, by Proposition 1.3.2, u_t is already continuous, and by Proposition 1.3.1, ∇u is globally defined in L^∞. We will prove that it is continuous at every point.

In the interior of the contact set, $u(x, t) \equiv \varphi(x) \in C^1$, and in the interior of $\{u > \varphi\}$, we can use interior estimates (Proposition 1.6.4) to see that u is C^1.

Therefore, we only need to work with the points on the free boundary. Assume without loss of generality that the origin is a free boundary point, and we will prove that ∇u is continuous at it.

Let $v = u - \varphi$. After a scaling and a translation, we can apply Lemma 1.3.3 to obtain

$$\sup_{B_R(x_0)\times(t_0 - R^{2s}, t_0]} |\nabla v| \leq CR^\tau,$$

for all $R \geq 0$. The constant C here depends only on φ, the dimension, s and the ellipticity constants.

We distinguish two cases:

Case 1. If *the free boundary continues to the future*, more precisely, for all $\rho \in (0, r)$, there exists $t_\rho > 0$ such that

$$\{v = 0\} \cap (B_\rho \times \{t_\rho\}) \neq \emptyset,$$

it follows that for all $t \in (0, t_\rho)$, $\{v = 0\} \cap (B_\rho \times \{t\}) \neq \emptyset$, because $u_t \geq 0$ and therefore the contact set shrinks in time.

Let $\delta \in (0, r)$. Let $|x| < \delta$, and $t < t_\delta$ as defined above. Then, there exists $x' \in B_\delta$ such that (x', t) belongs to the contact set, and it follows that

$$|\nabla v(x,t)| \leq C|x - x'|^\tau \leq C(2\delta)^\tau.$$

Then, letting $\delta \to 0$, we obtain a sequence of neighbourhoods of the origin where $|\nabla v| \leq C(2\delta)^\tau$, and hence ∇v vanishes continuously at $(0, 0)$.

Case 2. If *the free boundary ends at the origin*, there exists some $r_0 > 0$ such that for all $t > 0$, $v > 0$ in $B_{r_0} \times \{t\}$. Assume after a scaling that $r_0 = 1$ (notice that L may change but the ellipticity constants will be the same). We will prove that the limit of v_i is zero as it approaches the origin. If we approach from the past, then $(0, -t)$ belongs to the contact set for all $t > 0$, and we can use the same argument that in Case 1.

To consider approaching the origin from the future, recall that u solves $u_t = (Lu)^-$ globally, hence, we can consider u a solution of the nonlocal heat equation with right hand side

$$(\partial_t + L)u = (Lu)^+ \quad \text{in} \quad \mathbb{R}^n \times (0, T')$$

and apply Duhamel's formula at (x, t) with $x \in B_{1/2}$ and $t \in (0, \frac{1}{2})$, to get

$$u(x,t) = \int_{\mathbb{R}^n} p_t(x-y)u(y,0)\mathrm{d}y + \int_0^t \int_{\mathbb{R}^n} p_{t-\zeta}(x-y)(Lu)^+(y,\zeta)\mathrm{d}y\mathrm{d}\zeta,$$

where $p_t(x)$ is the fundamental solution for this particular operator (see Theorem 1.6.1). Then, differentiating with respect to x_i and using that $p_t \in C^\infty$ and u is Lipschitz,

$$u_i(x,t) = \int_{\mathbb{R}^n} p_t(x-y)u_i(y,0)\mathrm{d}y + \int_0^t \int_{\mathbb{R}^n} \partial_i p_{t-\zeta}(x-y)(Lu)^+(y,\zeta)\mathrm{d}y\mathrm{d}\zeta.$$

Now let us estimate both integrals separately. For the first one, we will use that $|u_i(y,0)| \leq C|y|^\tau$ by Lemma 1.3.3, as well as $|p_t(x)| \leq C \min\{t^{-\frac{n}{2s}}, t|x|^{-n-2s}\}$ by Theorem 1.6.1.

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} p_t(x-y)u_i(y,0)\mathrm{d}y\right| \lesssim \int_{\mathbb{R}^n} \min\{1, |y|^\tau\} \min\left\{t^{-\frac{n}{2s}}, \frac{t}{|x-y|^{n+2s}}\right\} \mathrm{d}y \\
&\lesssim \int_{B_{t^{\frac{1}{2s}}/2}(x)} t^{-\frac{n}{2s}}|y|^\tau \mathrm{d}y + \int_{B_{1/2}(x)\setminus B_{t^{\frac{1}{2s}}/2}(x)} \frac{t|y|^\tau}{|x-y|^{n+2s}}\mathrm{d}y + \int_{B^c_{1/2}(x)} \frac{t}{|x-y|^{n+2s}}\mathrm{d}y \\
&\leq t^{-\frac{n}{2s}}|x + t^{\frac{1}{2s}}|^\tau |B_{t^{\frac{1}{2s}}}| + t\int_{B_{1/2}\setminus B_{t^{\frac{1}{2s}}}} |y|^{-n-2s}|x+y|^\tau \mathrm{d}y + t\int_{B^c_{1/2}} |y|^{-n-2s}\mathrm{d}y \\
&\lesssim |x + t^{\frac{1}{2s}}|^\tau + t(t^{\frac{1}{2s}})^{-2s}|x|^\tau + t(t^{\frac{1}{2s}})^{\tau-2s} + t \lesssim t^{\frac{\tau}{2s}} + |x|^\tau.
\end{aligned}$$

For the second integral, we will use that Lu is bounded because u is Lipschitz, $Lu \leq 0$ in $B_{1/2}(x) \subset B_1$, as well as $Lu \leq 0$ outside of the support of the obstacle φ. Let R big enough such that $\operatorname{supp}\varphi \subset B_R$. Then, by Corollary 1.6.5,

$$\left|\int_0^t \int_{\mathbb{R}^n} \partial_i p_{t-\zeta}(x-y)(Lu)^+(y,\zeta)\mathrm{d}y\mathrm{d}\zeta\right| \lesssim \int_0^t \int_{B_R \setminus B_{1/2}(x)} |\nabla p_{t-\zeta}(x-y)|\mathrm{d}y\mathrm{d}\zeta$$
$$\lesssim \int_0^t \int_{B_R \setminus B_{1/2}(x)} 1\mathrm{d}y\mathrm{d}\zeta \lesssim t.$$

Therefore, $|u_i(x,t)| \lesssim t^{\frac{\tau}{2s}} + |x|^\tau$ for $t > 0$, and it converges to zero as it approaches the origin from the future, concluding that ∇u is continuous in x and t at that point. □

1.4 Optimal $C^{1,1}$ regularity

In this section, we establish the optimal $C^{1,1}$ regularity of solutions. First, we prove that the free boundary moves at a positive *speed*.

Proposition 1.4.1. *Let $s \in (0, \frac{1}{2})$, and let u be the solution of (1.1) with L an operator satisfying (1.2) and (1.3), and $\varphi \in C_c^{2,1}(\mathbb{R}^n)$[3]. Let $v = u - \varphi$, and let $0 < t_1 < t_2 < T$. Then,*

$$|\nabla v| \leq C v_t \quad in \quad \mathbb{R}^n \times [t_1, t_2],$$

for some positive C, depending only on t_1, t_2, φ, the dimension, s and the ellipticity constants.

Moreover, the free boundary is the graph of a Lipschitz function $\{t = \Gamma(x)\}$ in $\mathbb{R}^n \times (t_1, t_2)$, with the same Lipschitz constant C.

To prove this proposition, we will use the following positivity lemma, see [46, Lemma 6.2] for the elliptic version.

Lemma 1.4.2. *Let $E \subset Q_1$ be compact, let L be an operator satisfying (1.2) and (1.3), and let $w \in C(Q_1) \cap C^1(Q_1 \setminus E)$ satisfying*

$$\begin{cases} |\partial_t w + Lw| & \leq & \varepsilon & in & Q_1 \setminus E \\ w & = & 0 & in & E \\ w & \geq & -\varepsilon & in & E^c, \end{cases}$$

in the viscosity sense, and also

$$\int_{\mathbb{R}^n} \frac{w^+(x,t)}{1+|x|^{n+2s}}\mathrm{d}x \geq 1 \quad for\ all \quad t \in [-1,1].$$

Then,

$$w \geq 0 \quad in \quad \overline{Q_{1/2}}.$$

The constant $\varepsilon > 0$ depends only on s, the dimension and the ellipticity constants.

[3]The compactness of the support is a technical condition needed for the proof of this proposition but it does not seem crucial for the problem.

Proof. Let $\psi \in C_c^\infty(Q_{3/4})$, with $\psi \equiv 1$ in $Q_{1/2}$ and $0 \le \psi \le 1$. We proceed by contradiction. Suppose the lemma does not hold. Then, for some $c > 0$, the function

$$\psi_{\varepsilon,c} = -c - \varepsilon + \varepsilon\psi$$

touches w from below in $(x_0, t_0) \in Q_{3/4}$. Moreover, $(x_0, t_0) \in E^c$ because $w(x_0, t_0) < 0$, so $(x_0, t_0) \in Q_1 \setminus E$.

Now we compute $(\partial_t + L)w(x_0, t_0)$ to obtain a contradiction. By the definition of (x_0, t_0), $w - \psi_{\varepsilon,c}$ attains a global minimum there. Thus,

$$\begin{aligned}
(\partial_t + L)(w - \psi_{\varepsilon,c})(x,t) &= L(w - \psi_{\varepsilon,c})(x,t) \\
&= -\int_{\mathbb{R}^n} (w(x+y,t) - \psi_{\varepsilon,c}(x+y,t))K(y)\mathrm{d}y \\
&\le -\lambda \int_{\mathbb{R}^n} w^+(x+y,t)|y|^{-n-2s}\mathrm{d}y \\
&\le -\lambda \int_{\mathbb{R}^n} \frac{w^+(y,t)}{|y-x|^{n+2s}}\mathrm{d}y \le -C\lambda,
\end{aligned}$$

using that $\psi_{\varepsilon,c} < 0$ and that $|y-x|^{n+2s} \le C(1+|y|^{n+2s})$ for any $x \in B_{3/4}$, with C depending only on $n + 2s$.

On the other hand,

$$(\partial_t + L)(w - \psi_{\varepsilon,c})(x,t) = (\partial_t + L)w(x,t) - (\partial_t + L)\psi_{\varepsilon,c} \le \varepsilon + \varepsilon\|(\partial_t + L)\psi\|_{L^\infty(Q_{3/4})},$$

and choosing ε small enough we get a contradiction. □

Using this lemma we are now able to prove that the free boundary *moves at all values of* t, i.e., it is a Lipschitz graph in the t direction.

Proof of Proposition 1.4.1. We will prove the inequality for any directional derivative v_i instead of the gradient. The result follows as a consequence.

Let $R \ge \max\{1, T^{\frac{1}{2s}}\}$ be such that $\operatorname{supp}\varphi \subset B_R$ and let $P > 0$ large, to be chosen later. Consider the set $A = \overline{B_1}(3Re_1) \times [\frac{t_1}{2}, \frac{t_2+T}{2}]$. Then, by construction, $A \subset \{v > 0\}$, and from Lemma 1.2.3 and compactness, it follows that $v_t \ge a > 0$ in A.

Let $r > 0$ such that for all $(x_0, t_0) \in B_{PR} \times [t_1, t_2]$, $Q_r(x_0, t_0) \subset \mathbb{R}^n \times [\frac{t_1}{2}, \frac{t_2+T}{2}]$. We will use a rescaled Lemma 1.4.2 in $Q_r(x_0, t_0)$ with a suitable linear combination

$$w = Mv_t - mv_i$$

with some positive M and m to be chosen later.

First, let E be the contact set. Then, $w \ge -m\|v_i\|_{L^\infty(\mathbb{R}^n\times(0,T))} \ge -2m\|\varphi\|_{C^{0,1}(\mathbb{R}^n)}$ in the whole space by Proposition 1.3.1. Moreover, in E^c we have

$$|(\partial_t + L)w| = m|(\partial_t + L)v_i| = m| - L\varphi_i| \le m\|\varphi\|_{C^{1,1}(\mathbb{R}^n)} \quad \text{in} \quad E^c.$$

On the other hand, for all $t \in [t_0 - r^{2s}, t_0 + r^{2s}]$,

$$\int_{\mathbb{R}^n} \frac{w^+(x,t)}{1+|x-x_0|^{n+2s}}\mathrm{d}x \ge \int_{B_1(3Re_1)} \frac{w^+(x,t)}{1+|x-x_0|^{n+2s}}\mathrm{d}x \ge$$
$$\int_{B_1(3Re_1)} \frac{Ma - m\|v_i\|_{L^\infty(\mathbb{R}^n\times(0,T))}}{1+|x-x_0|^{n+2s}}\mathrm{d}x \ge \frac{(Ma - m\|\varphi\|_{C^{0,1}(\mathbb{R}^n)})|B_1|}{1+(PR+3R+1)^{n+2s}}.$$

Then, choosing m small enough and M big enough suffices to be able to apply Lemma 1.4.2, and these constants depend only n, s, λ, Λ, R and φ. Therefore, $w \geq 0$ in $B_{PR} \times [t_1, t_2]$.

Finally, outside of B_{PR}, we will use a barrier argument. Since $v_t > 0$ in the set $(\overline{B_{PR/2}} \setminus B_R) \times [0,T]$, by compactness we can choose M and m such that $w(\cdot,0) \geq 0$ in B_{PR} and also $w \geq m$ in $(\overline{B_{PR/2}} \setminus B_R) \times [0,T]$.

Let $\tilde{w} = w + m(1 + 2\|\varphi\|_{C^{0,1}(\mathbb{R}^n)})\chi_{B_R}$. Now, since

$$w \geq mv_i \geq -m\|v_i\|_{L^\infty(\mathbb{R}^n \times [0,T])} \geq -2m\|\varphi\|_{C^{0,1}(\mathbb{R}^n)},$$

$\tilde{w} \geq m$ in $B_{PR/2} \times [0,T)$. On the other hand, $v = u - \varphi$ is identically zero at $t = 0$ and $v_t(\cdot,0) = -L\varphi > 0$ outside of the support of φ, and hence $\tilde{w}(\cdot,0) \geq 0$ in B^c_{PR}.

To apply the comparison principle, we also need to compute the right hand side for $x \in B^c_{PR}$. Using that u is a solution of the nonlocal heat equation,

$$(\partial_t + L)\tilde{w} = (\partial_t + L)(w + m(1 + 2\|\varphi\|_{C^{0,1}(\mathbb{R}^n)})\chi_{B_R}) = mL[\varphi_i + (1 + 2\|\varphi\|_{C^{0,1}(\mathbb{R}^n)})\chi_{B_R}],$$

and since the expression inside of the brackets is supported in B_R, for all x such that $|x| \geq PR$,

$$\begin{aligned} |(\partial_t + L)\tilde{w}| &\leq C'mR^n\|\varphi_i + (1 + 2\|\varphi\|_{C^{0,1}(\mathbb{R}^n)})\chi_{B_R}\|_{L^\infty(B_R \times [0,T))}(|x| - R)^{-n-2s} \\ &\leq CmR^n|x|^{-n-2s} \quad \text{in} \quad B^c_{PR} \times [t_1, t_2], \end{aligned}$$

where C depends only on n, s, λ, Λ and φ.

Let now ψ be defined as the solution of

$$\begin{cases} (\partial_t + L)\psi &= [(\partial_t + L)\tilde{w}]\,\chi_{B^c_{PR}} \quad \text{in} \quad \mathbb{R}^n \times (0,T) \\ \psi &= \frac{m}{2}\chi_{B_{PR/2}} \quad \text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Then, $|(\partial_t + L)\psi| \leq CmR^n(PR)^{-n-2s} = CmP^{-n-2s}R^{-2s}$, and it follows that $(\partial_t + L)(\psi - CmP^{-n-2s}R^{-2s}t) \leq 0$. Therefore, since it is a subsolution for the nonlocal heat equation, applying the comparison principle[4] with a constant we deduce $\psi - CmP^{-n-2s}R^{-2s}t \leq \frac{m}{2}$ in $\mathbb{R}^n \times (0,T)$, and in particular $\psi \leq \frac{m}{2} + CmP^{-n-2s}R^{-2s}T$. Choosing P large enough, $\psi \leq m$ in $\mathbb{R}^n \times (0,T)$.

Now, we apply the comparison principle again. Notice that $\psi \leq \tilde{w}$ at $t = 0$ by construction, and that $(\partial_t + L)\psi = (\partial_t + L)\tilde{w}$ for all $(x,t) \in B^c_{PR} \times (0,T)$. Furthermore, $\psi \leq m \leq \tilde{w}$ in $B_{PR} \times (0,T)$. Therefore, $\psi \leq \tilde{w}$ in $\mathbb{R}^n \times (0,T)$.

Finally, let

$$\tilde{\psi}(x,t) = \frac{2}{m|B_1|}\psi\left(\frac{PR}{2}x, \left(\frac{PR}{2}\right)^{2s} t\right).$$

Then, $\tilde{\psi}(\cdot,0) = |B_1|^{-1}\chi_{B_1}$, so it is positive, supported in B_1 and $\|\tilde{\psi}(\cdot,0)\|_{L^1(B_1)} = 1$. Moreover,

$$|(\partial_t + L)\tilde{\psi}| \leq \frac{2CmR^n}{m|B_1|}\left(\frac{PR}{2}\right)^{2s}\left|\frac{PR}{2}x\right|^{-n-2s}\chi_{B^c_1} \leq C'P^{-n}|x|^{-n-2s}\chi_{B^c_1},$$

and if we take P large enough such that $C'P^{-n} < \delta$, from Proposition 1.6.6 we get that $\tilde{\psi} \geq 0$ in $B^c_2 \times (0, T(PR/2)^{-2s})$[5]. Then, $\tilde{w} \geq 0$ in $B^c_2 \times (0,T)$, and since $\tilde{w} = w$ in $B^c_{PR} \times (0,T)$ we obtain $w \geq 0$ in $B^c_{PR} \times (0,T)$, as we wanted to prove.

From the inequality $|\nabla v| \leq Cv_t$, it follows that the free boundary is a Lipschitz graph in the t direction with constant C. □

[4] Here, ψ can be defined with the Duhamel formula and the heat kernel introduced in Theorem 1.6.1, and the comparison principle follows from the positivity of the heat kernel.

[5] Here we need to choose P large enough to have $T(PR/2)^{-2s} < 1$.

Once we know that the free boundary is a Lipschitz graph in the direction of t, we can use barriers to gain insight on the boundary behaviour of v_t. We will prove first a Hopf-type estimate in the t direction. Here we use crucially the fact that the diffusion is supercritical, i.e. $s < \frac{1}{2}$.

Proposition 1.4.3. *Let $s \in (0, \frac{1}{2})$, and let u be the solution of (1.1) with L an operator satisfying (1.2) and (1.3), and $\varphi \in C_c^{2,1}(\mathbb{R}^n)$. Let $v = u - \varphi$, and let $0 < t_1 < t_2 < T$. Then, there exists $c_0 > 0$ such that for all free boundary points $(x_0, t_0) \in \mathbb{R}^n \times [t_1, t_2]$,*

$$v_t(x_0, t_0 + t) \geq c_0 t \quad \textit{for} \quad t \in (0, \delta),$$

where c_0 and δ are small positive constants depending only on t_1, t_2, T, φ, the dimension, s and the ellipticity constants.

Proof. Let $R \geq 1$ such that $\operatorname{supp} \varphi \subset B_R$. Then, consider the compact set

$$A = \overline{B_1}(3Re_1) \times \left[\frac{t_1}{2}, \frac{t_2 + T}{2}\right].$$

Then, by construction, $A \subset \{v > 0\}$, and from Lemma 1.2.3 and compactness, it follows that $v_t \geq a > 0$ in A.

By Proposition 1.4.1, there exists C_0 such that $|\nabla v| \leq C_0 v_t$ in $\mathbb{R}^n \times [\frac{t_1}{2}, \frac{t_2+T}{2}]$. Assume without loss of generality that $C_0 \geq 1$.

Now, there exists $r > 0$ such that for all $(x, t) \in \mathbb{R}^n \times [t_1, t_2]$,

$$Q_r(x, t) \subset \mathbb{R}^n \times \left(\frac{t_1}{2}, \frac{t_2 + T}{2}\right).$$

Let (x_0, t_0) be a free boundary point with $t_0 \in [t_1, t_2]$, and define the cone

$$\mathcal{C} = \{t_0 + 2C_0|x - x_0| < t < t_0 + r^{2s}\} \subset \mathbb{R}^n \times \left(\frac{t_1}{2}, \frac{t_2 + T}{2}\right).$$

Since C_0 is also the Lipschitz constant of the free boundary in $\mathbb{R}^n \times (\frac{t_1}{2}, \frac{t_2+T}{2})$, $\mathcal{C}$ is entirely *above* the free boundary, and $v > 0$ in $\mathcal{C}$. Then, it follows from Lemma 1.2.3 that $v_t > 0$ in $\mathcal{C}$ as well.

With this information, we can construct a subsolution in $\mathcal{C}$ to compare with v_t. Let us assume after a translation that $(x_0, t_0) = (0, 0)$. Let w defined in $\mathbb{R}^n \times [0, r^{2s}]$ as follows:

$$w(x, t) = c_0(t - 2C_0|x|)_+ + a\chi_{\tilde{A}}(x, t) = c_0(t - 2C_0|x|)_+ + a\chi_{\overline{B_1}(3Re_1 - x_0)}(x),$$

with $c_0 > 0$ to be chosen later.

Then, we need to check that $(\partial_t + L)w \leq 0$ in $\mathcal{C}$ and that $w \leq v_t$ in $(\mathbb{R}^n \times (0, r)) \setminus \mathcal{C}$. The latter follows by construction, because for any $(x, t) \in \mathbb{R}^n \times (0, r)$ that does not belong to $\mathcal{C}$, $t - 2C_0|x| < 0$ and then $w \equiv a\chi_{\overline{B_1}(3Re_1 - x_0)}(x)$ in the relevant set. Thus, recalling that $v_t \geq a$ in A, $w \leq v_t$ outside of the cone.

To check that w is a subsolution in $\mathcal{C}$, first notice that $w_t = c_0$ inside the cone. Then,

$$\begin{aligned} Lw(x, t) &\leq c_0 \|L(t - 2C_0|x|)_+\|_{L^\infty(\mathbb{R}^n \times (0, r))} + a(L\chi_{\overline{B_1}(3Re_1 - x_0)})(x) \\ &\leq C_1 C_0 c_0 - a \int_{\mathbb{R}^n} \chi_{\overline{B_1}(3Re_1 - x_0)}(y) K(y) \mathrm{d}y \leq C_1 C_0 c_0 - \frac{a\lambda |B_1|}{(4R + 1)^{n+2s}}, \end{aligned}$$

where we used that $|x_0| < R$, and it follows that

$$(\partial_t + L)w \leq c_0 + C_1 C_0 c_0 - C_2$$

and then choosing c_0 small enough suffices to have $(\partial_t + L)w \leq 0$.

Finally, by the comparison principle[6], $v_t \geq w$ in $\mathcal{C}$, and in particular $v_t(0,t) \geq c_0 t$ for $t \in [0, r^{2s})$, and undoing the translation,

$$v_t(x_0, t_0 + t) \geq c_0 t \quad \text{for} \quad t \in (0, r^{2s}),$$

and for all $(x_0, t_0) \in \partial\{u > \varphi\} \cap (\mathbb{R}^n \times [t_1, t_2])$, as we wanted to prove. □

Integrating the lower bound for v_t, we can obtain a quadratic nondegeneracy of v in the t direction.

Corollary 1.4.4. *Let $s \in (0, \frac{1}{2})$, and let u be the solution of (1.1) with L an operator satisfying (1.2) and (1.3), and $\varphi \in C_c^{2,1}(\mathbb{R}^n)$. Let $v = u - \varphi$, and let $(x_0, t_0) \subset \mathbb{R}^n \times [t_1, t_2]$ be a free boundary point. Then, there exists $c_0 > 0$ such that*

$$v(x_0, t_0 + r) \geq c_0 r^2$$

for all $r \in (0, \delta)$, where c_0 and δ are positive and depend only on φ, t_1, t_2, T, s, the dimension and the ellipticity constants.

Proof. Use Proposition 1.4.3 to see that $v_t(x_0, t_0 + r) \geq c_0 r$ for all $r \in (0, \delta)$. Then, since $v \in C^1$, we can recover the value of v integrating v_t and therefore we get $v(x_0, t_0 + r) \geq v(x_0, t_0) + c_0 r^2/2 = c_0 r^2/2$. Finally rename $c_0/2$ as c_0. □

The counterpart is an upper bound for the growth of v_t. Much like the Hopf-type estimate can be proved with a subsolution taking advantage of a future cone of positivity, the anti-Hopf-type estimate is proved with a supersolution that takes advantage of a past cone in the contact set. Again, here we use crucially that the diffusion is supercritical.

Proposition 1.4.5. *Let $s \in (0, \frac{1}{2})$, and let u be the solution of (1.1) with L an operator satisfying (1.2) and (1.3), and $\varphi \in C_c^{2,1}(\mathbb{R}^n)$. Let $v = u - \varphi$, and let $0 < t_1 < t_2 < T$. Then, there exists $M > 0$ such that for all free boundary points $(x_0, t_0) \in \mathbb{R}^n \times [t_1, t_2]$,*

$$v_t(x_0, t_0 + t) \leq Mt \quad \textit{for all} \quad t > 0,$$

where M depends only on φ, t_1, t_2, T, s, the dimension and the ellipticity constants.

Proof. By Proposition 1.4.1, there exists C_0 such that $|\nabla v| \leq C_0 v_t$ in $\mathbb{R}^n \times [\frac{t_1}{2}, \frac{t_2+T}{2}]$. Assume without loss of generality that $C_0 \geq 1$.

Now, there exists $r > 0$ such that for all $(x,t) \in \mathbb{R}^n \times [t_1, t_2]$,

$$Q_r(x,t) \subset \mathbb{R}^n \times \left(\frac{t_1}{2}, \frac{t_2 + T}{2}\right).$$

[6] Here, v_t and w are classical solutions and the comparison principle follows from the standard pointwise bounds. We shall use this feature again in subsequent arguments.

Let (x_0, t_0) be a free boundary point with $t_0 \in [t_1, t_2]$, and define the cone

$$\mathcal{C} = \{t_0 - r^{2s} < t < t_0 - 2C_0|x - x_0|\} \subset \mathbb{R}^n \times \left(\frac{t_1}{2}, \frac{t_2 + T}{2}\right).$$

Notice that this cone is *backwards*, whereas the cone defined in the proof of Proposition 1.4.3 was *forward*. Since C_0 is also the Lipschitz constant of the free boundary in $\mathbb{R}^n \times (\frac{t_1}{2}, \frac{t_2+T}{2})$, $\mathcal{C}$ is entirely *below* the free boundary, and then $v_t \equiv 0$ in $\mathcal{C}$.

Assume after a translation that $(x_0, t_0) = (0, 0)$. Now, we want to construct a supersolution in

$$\Omega_\rho = B_\rho \times (-\rho, \rho) \setminus \mathcal{C},$$

with $\rho \in (0, r)$ to be chosen later.

To do so, we introduce the auxiliary function $h(x, t) := \min\{4C_0 + 1, (t + |x|)_+\}$. First, we notice $\partial_t h \equiv 1$ in $\{h > 0\} \cap Q_1$ and estimate Lh as follows.

$$\|Lh\|_{L^\infty(\mathbb{R}^n \times \mathbb{R})} \leq C_1 \|h\|_{C^{0,1}(\mathbb{R}^n \times \mathbb{R})} = C_1.$$

Let now $h_\rho(x, t) = h(4C_0\rho^{-1}x, \rho^{-1}t)$. By the scaling of the equation (notice that the bound on Lh depends on the ellipticity constants but not on the particular operator),

$$(\partial_t + L)h_\rho \geq \rho^{-1} - C_1(4C_0)^{2s}\rho^{-2s} \geq 0 \quad \text{in} \quad \Omega_\rho,$$

provided that ρ is small enough. Notice that ρ depends only on t_1, t_2, T, the dimension, s and the ellipticity constants.

Finally, let us check that there exists $M > 0$ such that $v_t \leq Mh_\rho$ in Ω_ρ. To do so, we will check that $v_t \leq Mh_\rho$ in the parabolic boundary of Ω_ρ. Indeed, $v_t = 0 \leq Mh_\rho$ in $\mathcal{C}$ for any positive M.

On the other hand, if we choose $M = \|v_t\|_{L^\infty(\mathbb{R}^n \times (0,T))}$, for all $t \in [-\rho, \rho]$ and $x \notin B_\rho$,

$$h_\rho(x, t) = \min\{1, \rho^{-1}(t + 4C_0|x|)_+\} \geq \min\{1, \rho^{-1}(-\rho + 4C_0\rho)_+\} = 1,$$

and for all $x \in \overline{B_\rho \times (-\rho, \rho) \setminus \mathcal{C}}$, $|x| \geq \frac{\rho}{2C_0}$, and therefore

$$h_\rho(x, -\rho) = \min\{1, \rho^{-1}(-\rho + 4C_0|x|)_+\} \geq \min\{1, (-1 + 2)_+\} = 1.$$

Hence,

$$v_t \leq \|v_t\|_{L^\infty(\mathbb{R}^n \times (0,T))} = M = Mh_\rho(x, t)$$

in the whole parabolic boundary of Ω_ρ, and together with the fact that $(\partial_t + L)h_\rho \geq 0$ in Ω_ρ we can conclude that $v_t \leq Mh_\rho$ in Ω_ρ by the comparison principle.

In particular, for every free boundary point $(x_0, t_0) \in \mathbb{R}^n \times [t_1, t_2]$, we have

$$v_t(x_0, t_0 + t) \leq Mt \quad \text{for} \quad t \in (0, \rho),$$

with uniform M and ρ.

To conclude, observe that $v_t(x_0, t_0 + t) \leq \rho^{-1}\|v_t\|_{L^\infty(\mathbb{R}^n \times (0,T))}t$ for all $t \geq \rho$, completing the proof. □

Now, using the previous estimate and the semiconvexity, we are ready to prove the global $C^{1,1}$ regularity of the solutions.

Proposition 1.4.6. *Let $s \in (0, \frac{1}{2})$, and let u be the solution of (1.1) with L an operator satisfying (1.2) and (1.3), and $\varphi \in C_c^{2,1}(\mathbb{R}^n)$. Let $0 < t_1 < t_2 < T$. Then, there exists $C > 0$ such that*

$$\|D_x^2 u\|_{L^\infty(\mathbb{R}^n \times [t_1,t_2])} + \|\partial_t \nabla u\|_{L^\infty(\mathbb{R}^n \times [t_1,t_2])} + \|\partial_{tt} u\|_{L^\infty(\mathbb{R}^n \times [t_1,t_2])} \leq C.$$

The constant C depends only on φ, t_1, t_2, T, s, the dimension and the ellipticity constants.

Proof. By Proposition 1.4.1, there exists $\eta \in (0,1)$ such that $\eta|\nabla v| \leq v_t$ in $\mathbb{R}^n \times [\frac{t_1}{3}, \frac{t_2+2T}{3}]$. Let e be a vector in the x directions with $|e| \leq 1$, and let $\nu = e_{n+1} + \eta e$. Thus, $\partial_\nu v = (\partial_t + \eta\partial_e)v \geq 0$ in $\mathbb{R}^n \times [\frac{t_1}{3}, \frac{t_2+2T}{3}]$.

Besides, for any given $(x,t) \in \mathbb{R}^n \times (0,T)$ and $r \in (0, 2^{-1-\frac{1}{2s}}t)$, consider the cutoff $\psi \in C_c^\infty(Q_{2^{1+\frac{1}{2s}}r}(x,t))$ with $\psi \equiv 1$ in $Q_{2^{\frac{1}{2s}}r}(x,t)$. By Proposition 1.2.4, since $|\nu| \leq \sqrt{2}$, $v_{\nu\nu} \geq -2\hat{C}$, and $\hat{C}$ does not depend on the choice of ν. Then,

$$0 \leq \int_{Q_{2^{\frac{1}{2s}}r}(x,t)} v_{\nu\nu} + 2\hat{C} \leq \int_{Q_{2^{1+\frac{1}{2s}}r}(x,t)} (v_{\nu\nu} + 2\hat{C})\psi = \int_{Q_{2^{1+\frac{1}{2s}}r}(x,t)} v\psi_{\nu\nu} + 2\hat{C}\psi \leq C(r),$$

and then $\|v_{\nu\nu}\|_{L^1(Q_{2^{\frac{1}{2s}}r}(x,t))} \leq C(r) + 2\hat{C}|Q_{2^{\frac{1}{2s}}r}| =: C_1(r)$. Observe that this bound is independent of (x,t) and ν.

Then we define the auxiliary function

$$w := \frac{\partial_\nu v(x + \eta he, t+h) - \partial_\nu v(x,t)}{h} = \frac{1}{h}\int_0^h \partial_{\nu\nu} v(x + \eta\zeta e, t+\zeta)\mathrm{d}\zeta.$$

Since w is an average of $v_{\nu\nu}$, we can obtain a L^1 bound as well. Let $h \in (0,r)$. Then,

$$\|w\|_{L^1(Q_r(x,t))} \leq \frac{1}{h}\int_0^h \|v_{\nu\nu}\|_{L^1(Q_r(x+\eta\zeta e, t+h))}\mathrm{d}\zeta \leq \|v_{\nu\nu}\|_{L^1(Q_{2^{\frac{1}{2s}}r}(x,t))} = C_1(r).$$

This shows that $w \in L^1((t_3,t_4] \to L^1_s(\mathbb{R}^n))$ for any $t_3, t_4 \in (0, T-h]$. Let us compute it: Let $r \in (0, 2^{-1-\frac{1}{2s}}t_3)$ and $N = \lceil \frac{t_4 - t_3}{2r} \rceil$. Then, we decompose the space in the following way:

$$\begin{aligned}
&\|w\|_{L^1((t_3,t_4]\to L^1_s(\mathbb{R}^n))} \leq \sum_{i=0}^{N-1} \|w\|_{L^1((t_3+2ir, t_3+2(i+1)r] \to L^1_s(\mathbb{R}^n))} + \|w\|_{L^1((t_4-2r,t_4]\to L^1_s(\mathbb{R}^n))} \\
&= \sum_{i=0}^{N-1} \int_{t_3+2ir}^{t_3+2(i+1)r} \int_{\mathbb{R}^n} \frac{|w(x,t)|}{1+|x|^{n+2s}}\mathrm{d}x\mathrm{d}t + \int_{t_4-2ir}^{t_4}\int_{\mathbb{R}^n} \frac{|w(x,t)|}{1+|x|^{n+2s}}\mathrm{d}x\mathrm{d}t \\
&\leq \sum_{i=0}^{N-1} \sum_{x\in\mathbb{Z}^n} \int_{t_3+2ir}^{t_3+2(i+1)r} \int_{B_r(rx/\sqrt{n})} \frac{|w(x,t)|}{1+|x|^{n+2s}}\mathrm{d}x\mathrm{d}t \\
&\quad + \sum_{x\in\mathbb{Z}^n} \int_{t_4-2ir}^{t_4} \int_{B_r(rx/\sqrt{n})} \frac{|w(x,t)|}{1+|x|^{n+2s}}\mathrm{d}x\mathrm{d}t \\
&= \sum_{i=0}^{N-1} \sum_{x\in\mathbb{Z}^n} \int_{Q_r(rx/\sqrt{n}, t_3+(2i+1)r)} \frac{|w(x,t)|}{1+|x|^{n+2s}}\mathrm{d}x\mathrm{d}t + \sum_{x\in\mathbb{Z}^n} \int_{Q_r(rx/\sqrt{n}, t_4-r)} \frac{|w(x,t)|}{1+|x|^{n+2s}}\mathrm{d}x\mathrm{d}t \\
&\leq N \sum_{x\in\mathbb{Z}^n} \frac{C_1(r)}{1+(|rx/\sqrt{n}|-r)_+^{n+2s}} =: NC_2(r).
\end{aligned}$$

Moreover, let $\tau_y w$ be the translation of w by the vector $y \in \mathbb{R}^n$. Analogously, we can deduce that

$$\|\tau_y w\|_{L^1((t_3,t_4]\to L^1_s(\mathbb{R}^n))} \leq NC_2(r),$$

independently of y.

Now, recall that v is a solution of $(\partial_t + L)v = -L\varphi$ in the set $\{v > 0\}$. Furthermore, if $v > 0$ at $(x,t) \in \mathbb{R}^n \times [\frac{t_1}{3}, \frac{t_2+2T}{3}]$, since $\partial_\nu v \geq 0$, $v(x + \eta h e, t + h) > 0$ also holds (provided that $t + h \leq \frac{t_2+2T}{3}$), and it follows that the translated function is also a solution. Hence,

$$\partial_t w + Lw = \eta \frac{\partial_e L\varphi(x) - \partial_e L\varphi(x + \eta h e)}{h} \quad \text{in} \quad \{v > 0\} \cap \left(\mathbb{R}^n \times \left[\frac{t_1}{3}, \frac{t_2+2T}{3} - h\right]\right),$$

and then $|\partial_t w + Lw| \leq C\|L\varphi\|_{C^{1,1}(\mathbb{R}^n)} \leq C\|\varphi\|_{C^{2,1}(\mathbb{R}^n)}$ in

$$\{v > 0\} \cap \left(\mathbb{R}^n \times \left[\frac{t_1}{3}, \frac{t_2+2T}{3} - h\right]\right) \subset \{v > 0\} \cap \left(\mathbb{R}^n \times \left[\frac{2t_1}{3}, \frac{2t_2+T}{3}\right]\right),$$

provided that h is small enough.

Moreover, if $(x_1,t_1) \in \{v = 0\} \cap (\mathbb{R}^n \times [\frac{2t_1}{3}, \frac{2t_2+T}{3}])$, then $\partial_\nu v(x_1,t_1) = 0$, and using Proposition 1.4.5 and taking h small enough, it follows that

$$\begin{aligned} w(x_1,t_1) &= \frac{\partial_\nu v(x_1 + \eta h e, t_1 + h)}{h} \leq \frac{2v_t(x_1 + \eta h e, t_1 + h)}{h} \\ &\leq \frac{2M(t_1 + h - \Gamma(x_1 + \eta h e))_+}{h} \leq \frac{2M(h + C_0 \eta h|e| + t_1 - \Gamma(x_1))}{h} \\ &\leq 4M. \end{aligned}$$

Therefore, $\tilde{w} = \max\{w, 4M\}$ is a subsolution for

$$\partial_t \tilde{w} + L\tilde{w} \leq C\|\varphi\|_{C^{2,1}(\mathbb{R}^n)} \quad \text{in} \quad \mathbb{R}^n \times \left[\frac{2t_1}{3}, \frac{2t_2+T}{3}\right],$$

and we can apply Lemma 1.6.3 to $\tau_y \tilde{w}$ obtain

$$\sup_{B_1 \times [t_1,t_2]} \tau_y \tilde{w} \leq C\left(\|\tau_y \tilde{w}\|_{L^1\left(\left(\frac{2t_1}{3}, \frac{2t_2+T}{3}\right] \to L^1_s(\mathbb{R}^n)\right)} + \|\varphi\|_{C^{2,1}(\mathbb{R}^n)}\right),$$

with C depending only on t_1, t_2, T, the dimension, the ellipticity constants and s. Then, since the bound is uniform on y, it follows from the definition of w that

$$\sup_{\mathbb{R}^n \times [t_1,t_2]} w \leq C(NC_2(r) + 2M + \|\varphi\|_{C^{2,1}(\mathbb{R}^n)}) =: C_0.$$

Since C_0 does not depend on ν or h, combining this with Proposition 1.2.4, it follows that $\|v_{\nu\nu}\|_{L^\infty(\mathbb{R}^n \times [t_1,t_2])} \leq C_* = \max\{C_0, 2\hat{C}\}$ for all $\nu = e_{n+1} + \eta e$ with e in the x direction and $|e| < 1$.

Now, let $e = \lambda \hat{e}$ with $\hat{e}$ a unit vector. Then,

$$D^2_{e_{n+1}+\eta e} v = v_{tt} + \eta(v_{te} + v_{et}) + \eta^2 v_{ee} = v_{tt} + \eta\lambda(v_{t\hat{e}} + v_{\hat{e}t}) + \eta^2\lambda^2 v_{\hat{e}\hat{e}}.$$

Since this expression is bounded by C_* for all values of $\hat{e}$ and $\lambda \in (-1,1)$, we can evaluate at $\lambda = 0, \frac{1}{2}, -\frac{1}{2}$ to get:

$$\begin{aligned} &|v_{tt}| \le C_* \\ &\left| v_{tt} + \frac{1}{2}\eta(v_{t\hat{e}} + v_{\hat{e}t}) + \frac{1}{4}\eta^2 v_{\hat{e}\hat{e}} \right| \le C_* \\ &\left| v_{tt} - \frac{1}{2}\eta(v_{t\hat{e}} + v_{\hat{e}t}) + \frac{1}{4}\eta^2 v_{\hat{e}\hat{e}} \right| \le C_*, \end{aligned}$$

and then it is easy to check that $|v_{\hat{e}\hat{e}}| + |v_{t\hat{e}} + v_{\hat{e}t}| \le C(\eta)C_*$.

Hence, for any $e \in \mathbb{S}^n$ (all unit vectors in x, t), $|v_{ee}| \le C'(\eta)C_*$. Then, given two points (x_1, t_1) and (x_2, t_2) in $\mathbb{R}^n \times [t_1, t_2]$,

$$|v(x_1,t_1) - \nabla_{x,t}v(x_1,t_1)\cdot(x_2-x_1,t_2-t_1) - v(x_2,t_2)| \le C'(\eta)C_*\|(x_1-x_2,t_1-t_2)\|^2.$$

This means that $v \in C^{1,1}(\mathbb{R}^n \times [t_1,t_2])$, and $u = v + \varphi$ as well. □

We can now give the:

Proof of Theorem 1.1.1. The global Lipschitz regularity follows from Proposition 1.3.1. The $C^{1,1}$ regularity follows from Proposition 1.4.6. □

1.5 Regularity of the free boundaries

In this section we use the regularity of the solutions established before to deduce the regularity of the free boundaries. Here again, we will use crucially the fact that $s < \frac{1}{2}$. We first take advantage of the different orders of derivation in the equation (1.1) to obtain further regularity in t.

Lemma 1.5.1. *Let $s \in (0, \frac{1}{2})$, let u be the solution of (1.1) with L an operator satisfying (1.2) and (1.3), and $\varphi \in C^{2,1}_c(\mathbb{R}^n)$. Let $v = u - \varphi$, and let $0 < t_1 < t_2 < T$. Then, there exists $C > 0$ such that*

$$\|v_{tt}\|_{C^\alpha((\mathbb{R}^n\times[t_1,t_2])\cap\{v>0\})} + \sum_{i=0}^{n} \|v_{ti}\|_{C^\alpha((\mathbb{R}^n\times[t_1,t_2])\cap\{v>0\})} \le C.$$

where $\alpha = 1 - 2s > 0$.

Proof. Let $\nu \in \mathbb{S}^n$ be any unit vector in x and t, and let $w = \partial_\nu u$. Then, by Proposition 1.4.6, $\|w\|_{C^{0,1}(\mathbb{R}^n\times[t_1,t_2])} \le C$. Moreover, by the same arguments as in the proof of Proposition 1.3.2, we deduce $\|Lw\|_{C^\alpha(\mathbb{R}^n\times[t_1,t_2])} \le C$.

Then, since $v_t = u_t = -Lu$ in $\{v > 0\}$, differentiating the equation with respect to ν it follows that $w_t = -Lw$ in $\{v > 0\}$, and therefore $\|v_{t\nu}\|_{C^\alpha(\mathbb{R}^n\times[t_1,t_2])} \le C$. □

We next show that the free boundary is $C^{1,\alpha}$.

Theorem 1.5.2. *Let $s \in (0, \frac{1}{2})$, and let u be the solution of (1.1) with L an operator satisfying (1.2) and (1.3), and $\varphi \in C^{2,1}_c(\mathbb{R}^n)$. Let $0 < t_1 < t_2 < T$.*

Then, the free boundary is a $C^{1,\alpha}$ graph in the t direction in $\mathbb{R}^n \times [t_1, t_2]$, i.e.

$$\partial\{u > \varphi\} \cap (\mathbb{R}^n \times (t_1, t_2)) = \{t = \Gamma(x)\},$$

with $\Gamma \in C^{1,\alpha}$ and $\alpha = 1 - 2s > 0$.

Proof. We already know that the free boundary is a Lipschitz graph by Proposition 1.4.1. Then, let $\alpha = 1 - 2s$. By Lemma 1.5.1,

$$\|v_{tt}\|_{C^\alpha((\mathbb{R}^n\times[t_1,t_2])\cap\{v>0\})} + \sum_{i=0}^{n} \|v_{ti}\|_{C^\alpha((\mathbb{R}^n\times[t_1,t_2])\cap\{v>0\})} \leq C.$$

Then, $v_{tt} > 0$ at the free boundary by Proposition 1.4.3, and by continuity $v_{tt} \geq c_0$ in $E = \{t \in [\Gamma(x), \Gamma(x) + \delta]\} \cap [t_1, t_2]$ for some small $\delta > 0$. Thus,

$$\left\| \frac{v_{ti}}{v_{tt}} \right\|_{C^\alpha(E)} \leq C.$$

Finally, notice that the free boundary can be seen as the zero level surface of v_t. The normal vector to the level surfaces of v_t is given by the formula

$$\nu = \frac{\nabla_{x,t} v_t}{|\nabla_{x,t} v_t|} = \frac{(\partial_{t1}v/\partial_{tt}v, \ldots, \partial_{tn}v/\partial_{tt}v, 1)}{\sqrt{1 + \sum_{j=1}^{n} (\partial_{tj}v/\partial_{tt}v)^2}},$$

and therefore $\nu \in C^\alpha(E)$ uniformly, thus $\{v_t = 0\}$ is a $C^{1,\alpha}$ manifold, as desired. □

Once we know that the free boundary is a $C^{1,\alpha}$ graph, we can provide an expansion for the solution.

Corollary 1.5.3. *Let $s \in (0, \frac{1}{2})$, and let u be the solution of (1.1) with L an operator satisfying (1.2) and (1.3), and $\varphi \in C_c^{2,1}(\mathbb{R}^n)$. Let $(x_0, t_0) \in \partial\{u > \varphi\}$ be a free boundary point. Then,*

$$u_t(x_0 + x, t_0 + t) = c_0(t - a \cdot x)_+ + O(t^{1+\alpha} + |x|^{1+\alpha})$$

and

$$(u - \varphi)(x_0 + x, t_0 + t) = \frac{c_0}{2}(t - a \cdot x)_+^2 + O(t^{2+\alpha} + |x|^{2+\alpha}),$$

with $\alpha = 1 - 2s > 0$, $c_0 = u_{tt}(x_0, t_0) > 0$ and $a = \nabla\Gamma(x_0)$.

Proof. We will use strongly that $\Gamma \in C^{1,\alpha}$ by Theorem 1.5.2, and that $u_t \in C^{1,\alpha}(\overline{\{u > \varphi\}})$ by Lemma 1.5.1.

We distinguish two cases. If $(x_0 + x, t_0 + t) \in \{u = \varphi\}$, $t_0 + t \leq \Gamma(x_0 + x)$, then expanding $\Gamma(x_0 + x) = t_0 + \nabla\Gamma(x_0) \cdot x + O(|x|^{1+\alpha})$ we obtain

$$t - \nabla\Gamma(x_0) \cdot x \leq O(|x|^{1+\alpha}),$$

and therefore

$$(t - \nabla\Gamma(x_0) \cdot x)_+^2 \leq O(|x|^{2+2\alpha}) \leq O(|x|^{2+\alpha}),$$

and since $(u - \varphi)(x_0 + x, t_0 + t) = u_t(x_0 + x, t_0 + t) = 0$ this is exactly what we needed.

On the other hand, outside of the contact set,

$$\begin{aligned} u_t(x_0 + x, t_0 + t) &= \int_{\Gamma(x_0+x)}^{t_0+t} u_{tt}(x_0 + x, \tau) d\tau \\ &= (t_0 + t - \Gamma(x_0 + x))(u_{tt}(x_0, t_0) + O(t^\alpha + |x|^\alpha)) \\ &= (t - \nabla\Gamma(x_0) \cdot x)_+ u_{tt}(x_0, t_0) + O(t^{1+\alpha} + |x|^{1+\alpha}), \end{aligned}$$

where in the last equality we expanded $\Gamma(x_0+x)$ as before, and if $t-\nabla\Gamma(x_0)\cdot x \le 0$, the whole term is $O(t^{1+\alpha}+|x|^{1+\alpha})$ and can be absorbed in the error term because $t_0+t-\Gamma(x_0+x)\ge 0$.

Then, we can repeat the procedure and integrate u_t, knowing already its expansion, and the conclusion follows from an analogous computation. □

We can now give the:

Proof of Theorem 1.1.2. The first part is Theorem 1.5.2, the second part is Corollary 1.5.3. □

1.5.1 Regular and singular points

Definition 1.5.4. Let $s\in(0,\frac{1}{2})$, and let u be the solution of (1.1) with L an operator satisfying (1.2) and (1.3), and $\varphi \in C_c^{2,1}(\mathbb{R}^n)$. Let $(x_0,t_0)\in\partial\{u>\varphi\}$ be a free boundary point. Then,

- We say (x_0,t_0) is a *regular* free boundary point if there exists $c_0>0$ such that for all small $r>0$,
$$\|u(\cdot,t_0)-\varphi\|_{L^\infty(B_r(x_0))}\ge c_0 r^2.$$

- We say (x_0,t_0) is a *singular* free boundary point if it is not regular.

One important first observation is the following.

Proposition 1.5.5. *Let $s\in(0,\frac{1}{2})$, and let u be the solution of (1.1) with L an operator satisfying (1.2) and (1.3), and $\varphi\in C_c^{2,1}(\mathbb{R}^n)$. Then, if (x_0,t_0) is any free boundary point, the following are equivalent:*

i. (x_0,t_0) is a regular free boundary point.

ii. If ν_0 is the normal vector to the free boundary at (x_0,t_0), $\nu_0\neq e_{n+1}$.

iii. $\nabla u_t(x_0,t_0)\neq 0$.

Moreover, the set of regular free boundary points is an open subset of $\partial\{u>\varphi\}$.

Proof. (ii) ⇔ (iii):

It follows directly from

$$\nu_0=\frac{(\nabla u_t(x_0,t_0),u_{tt}(x_0,t_0))}{\sqrt{1+|\nabla u_t(x_0,t_0)|^2/u_{tt}(x_0,t_0)^2}}$$

and the fact that $u_{tt}(x_0,t_0)>0$.

(i) ⇔ (ii):

We will distinguish the cases $\nu_0=e_{n+1}$ and $\nu_0\neq e_{n+1}$. If $\nu_0=e_{n+1}$, let $\{t=\Gamma(x)\}$ be the free boundary. Then, $\Gamma\in C^{1,\alpha}$ and $\nabla\Gamma(x_0)=0$ because $\nu_0=e_{n+1}$. Then,

$$\Gamma(x_0+x)\ge t_0-C|x|^{1+\alpha},$$

and therefore

$$\begin{aligned}(u-\varphi)(x_0+x,t_0) &\leq \int_{\Gamma(x_0+x)}^{t_0}\int_{\Gamma(x_0+x)}^{\tau} u_{tt}\mathrm{d}\tau'\mathrm{d}\tau\\ &\leq \frac{(t_0-\Gamma(x_0+x))^2}{2}\|u_{tt}\|_{L^\infty(\mathbb{R}^n\times[\Gamma(x_0+x),t_0])}\\ &\leq C|x|^{2+2\alpha},\end{aligned}$$

contradicting the assumption that (x_0,t_0) is a regular point.

On the other hand, if $\nu_0=\alpha e_{n+1}+\beta e$, with e a unit vector in the x directions and $\beta>0$, we can also approximate Γ as

$$\Gamma(x_0+x)\leq t_0-\frac{\beta}{\alpha}(x\cdot e)+C|x|^{1+\alpha}.$$

Notice that $\alpha\neq 0$ because $u_{tt}>0$ on the free boundary as a consequence of Proposition 1.4.3. We also need to use that, for some small $\delta>0$, $u_{tt}\geq c_\delta>0$ in the set $E_\delta=\{t\in[\Gamma(x),\Gamma(x)+\delta]\}\cap[t_0-\delta,t_0+\delta]$, by the same argument as in the proof of Theorem 1.5.2.

Then, if r is small,

$$\|u(\cdot,t_0)-\varphi\|_{L^\infty(B_r(x_0))}\geq u(x_0+\frac{r}{2}e,t_0)-\varphi(x_0+\frac{r}{2}e)\geq\frac{1}{2}\left(\frac{\beta r}{2\alpha}-Cr^{1+\alpha}\right)^2 c_\delta\geq c_0r^2.$$

For the last part, first notice that ∇u_t is a continuous function in $\overline{\{u>\varphi\}}$ because $u_t\in C^{1,\alpha}(\{u>\varphi\})$ by Lemma 1.5.1. As a consequence, the set of regular points, $\{\nabla u_t\neq 0\}\cap\partial\{u>\varphi\}$, is a relatively open set. □

In a neighbourhood of a regular free boundary point, the free boundary is also $C^{1,\alpha}$ in space:

Proposition 1.5.6. *Let $s\in(0,\frac{1}{2})$, and let u be the solution of (1.1) with L an operator satisfying (1.2) and (1.3), and $\varphi\in C_c^{2,1}(\mathbb{R}^n)$. Let (x_0,t_0) be any regular free boundary point.*

Then, there exists an open neighbourhood $x_0\in U\subset\mathbb{R}^n\times(0,T)$ such that the free boundary is a $C^{1,\alpha}$ graph in the x direction, i.e., there exists $i\in\{0,\dots,n\}$ such that

$$\partial\{u>\varphi\}\cap U=\{x_i=F_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n,t)\},$$

with $F_i\in C^{1,\alpha}$ and $\alpha=1-2s>0$.

Proof. First, by Theorem 1.5.2, the free boundary can be represented as $\partial\{u>\varphi\}=\{t=\Gamma(x)\}$ in a neighbourhood of (x_0,t_0), with $\Gamma\in C^{1,\alpha}$. Moreover, since (x_0,t_0) is regular, by Proposition 1.5.5, the normal vector to the free boundary $\nu_{(x_0,t_0)}\neq e_{n+1}$, and thus $\nabla\Gamma(x_0)\neq 0$, and in particular $\partial_{x_i}\Gamma(x_0)\neq 0$.

Therefore, by the implicit function theorem, $\{u>\varphi\}\cap\{t=t_0\}$ is locally a $C^{1,\alpha}$ graph of the form $(x_1\dots,x_{i-1},x_{i+1},\dots,x_n,t)\mapsto x_i$. □

On the other hand, in the time slice of a singular point, the free boundary could be very complicated. Nevertheless, we can prove that singular points are scarce. To do so, we will use the following lemma from geometric measure theory.

Lemma 1.5.7 ([102]). *Consider the family* $\{E_t\}_{t\in(0,T)}$ *with* $E_t \subset \mathbb{R}^n$*, and let us denote* $E := \bigcup_{t\in(0,T)} E_t$.

Let $1 \le \gamma \le \beta \le n$*, and assume that the following holds:*

- $\dim_{\mathcal{H}} E_t \le \beta$,
- *for all* $\varepsilon > 0$, $t_0 \in (0,T)$ *and* $x_0 \in E_{t_0}$*, there exists* $\rho > 0$ *such that*

  $$B_r(x_0) \cap E_t = \emptyset,$$

  for all $r \in (0,\rho)$ *and* $t > t_0 + r^{\gamma-\varepsilon}$.

Then, $\dim_{\mathcal{H}} E_t \le \beta - \gamma$*, for* $\mathcal{H}^1$*-a.e.* $t \in (0,T)$.

Using the global $C^{1,\alpha}$ regularity of the free boundary, and noticing that the normal vector is e_{n+1} at singular points, we can prove the following dimension bound.

Proposition 1.5.8. *Let* $s \in (0, \frac{1}{2})$*, and let* u *be the solution of (1.1) with* L *an operator satisfying (1.2) and (1.3), and* $\varphi \in C_c^{2,1}(\mathbb{R}^n)$*. Let* $\Sigma \subset \partial\{u > \varphi\}$ *be the set of singular free boundary points, and let* $\Sigma_t = \{(x,t') \in \Sigma : t' = t\}$ *be the time slices of the singular set.*

Then,

$$\dim_{\mathcal{H}} \Sigma_t \le n - 1 - \alpha, \quad \text{for almost every} \quad t \in (0,T),$$

with $\alpha = 1 - 2s > 0$*. In particular,* $\mathcal{H}^{n-1}(\Sigma_t) = 0$ *for almost every* $t \in (0,T)$.

Proof. We just need to check the hypotheses of Lemma 1.5.7, with $\beta = n$ and $\gamma = 1 + \alpha$. The first condition is obvious, because since $\Sigma_t \subset \mathbb{R}^n \times \{t\}$, $\dim_{\mathcal{H}} \Sigma_t \le n$.

For the second condition, we use the $C^{1,\alpha}$ regularity of the free boundary. Let $x_0 \in E_{t_0}$. This means that (x_0, t_0) is a singular free boundary point. In particular, since $v_t(x_0, \Gamma(x_0)) = 0$ and $v_{tt}(x_0, t_0) \ne 0$,

$$\nabla\Gamma(x_0) = -\frac{\nabla v_t(x_0,t_0)}{v_{tt}(x_0,t_0)} = 0.$$

Now, $\Gamma \in C^{1,\alpha}$. Therefore, $\Gamma(x) \le t_0 + C|x - x_0|^{1+\alpha}$ for all $x \in B_\rho(x_0)$ for some $\rho > 0$.

Finally, for any $\varepsilon > 0$, there exists $\rho(\varepsilon)$ such that for all $r \in (0, \rho(\varepsilon))$,

$$\Gamma(x) \le t_0 + Cr^{1+\alpha} < t_0 + r^{1+\alpha-\varepsilon},$$

and thus $B_r(x_0) \cap \Sigma_t = \emptyset$ for all $t > t_0 + r^{1+\alpha-\varepsilon}$, completing the proof. □

We finally give the:

Proof of Theorem 1.1.3. The first part follows from Proposition 1.5.5, the second is Proposition 1.5.6 and the last is Proposition 1.5.8. □

1.6 Appendix: Some tools for nonlocal parabolic equations

We start recalling the following estimates on the fundamental solution to the nonlocal heat equation, see [58].

Theorem 1.6.1 ([58]). *Let L be an operator satisfying (1.2) and (1.3), and let $w \in L^\infty(\mathbb{R}^n \times (0,T))$ be the solution of*

$$\begin{cases} (\partial_t + L)w = 0 & \text{in} \quad \mathbb{R}^n \times (0,T) \\ w = w_0 & \text{on} \quad \{t = 0\}. \end{cases}$$

Then,

$$w(x,t) = p_t * w_0,$$

and p_t is nonnegative, $\|p_t(\cdot,t)\|_{L^1(\mathbb{R}^n)} = 1$ for all $t \in (0,T)$,

$$(\partial_t + L)p_t = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,T),$$

and

$$c_1 \min\{t^{-\frac{n}{2s}}, t|x|^{-n-2s}\} \le p_t(x) \le c_2 \min\{t^{-\frac{n}{2s}}, t|x|^{-n-2s}\},$$

for some $0 < c_1 < c_2$ depending only on T, the dimension, s and the ellipticity constants.

It is worth noticing that p_t is an approximation to the identity, in the following sense.

Corollary 1.6.2. *Let $f \in L^\infty(\mathbb{R}^n)$ be uniformly continuous, and define $f_t = p_t * f$ for all $t > 0$, with p_t the fundamental solution introduced in Theorem 1.6.1. Then,*

$$\|f_t\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^\infty(\mathbb{R}^n)}$$

and

$$\|f_t - f\|_{L^\infty(\mathbb{R}^n)} \to 0 \quad \text{as} \quad t \to 0.$$

Proof. Since $p_t \ge 0$ and $\|p_t(\cdot,t)\|_{L^1(\mathbb{R}^n)} = 1$, the trivial bound of the convolution suffices to obtain the first inequality.

For the second inequality, for any $\varepsilon > 0$ and any $x \in \mathbb{R}^n$,

$$\begin{aligned} |f_t(x) - f(x)| &= \left| \int_{\mathbb{R}^n} p_t(y)(f(x-y) - f(x))\mathrm{d}y \right| \\ &\le \int_{B_\delta} p_t(y)|f(x-y) - f(x)|\mathrm{d}y + \int_{B_\delta^c} p_t(y)|f(x-y) - f(x)|\mathrm{d}y \\ &\le \varepsilon \int_{B_\delta} p_t + 2\|f\|_{L^\infty(\mathbb{R}^n)} \int_{B_\delta^c} p_t \le \varepsilon + 2c_2\delta^{-2s}\|f\|_{L^\infty(\mathbb{R}^n)} t < 2\varepsilon, \end{aligned}$$

as we can choose δ sufficiently small to ensure $|f(x-y) - f(x)| < \varepsilon$ inside B_δ by uniform continuity, and then use Theorem 1.6.1 and make t tend to zero. □

We will also use the following L^1 to L^∞ bound for subsolutions.

Lemma 1.6.3. *Let L be an operator satisfying (1.2) and (1.3), and let $w \in L^\infty(\mathbb{R}^n \times (-1,0))$ be a subsolution of*

$$(\partial_t + L)w \le C_0 \quad \text{in} \quad \mathbb{R}^n \times (-1,0).$$

Then,

$$\sup_{B_1\times[-1+\delta,0)} w \le C\left(\int_{-1}^{0}\int_{\mathbb{R}^n} \frac{|w(x,t)|}{1+|x|^{n+2s}}\mathrm{d}x\mathrm{d}t + C_0\right),$$

where C depends only on $\delta > 0$, s, the dimension and the ellipticity constants.

Proof. Since $w - C_0(t+1) \ge w - C_0$ and $(\partial_t + L)(w - C_0(t+1)) \le 0$, we can assume without loss of generality that $C_0 = 0$.

Then, since w is a subsolution for the nonlocal heat equation, the following holds for any $-1 < t_0 < t < 0$:

$$w(x,t) \le \int_{\mathbb{R}^n} p_{t-t_0}(x-y)w(y,t_0)\mathrm{d}y,$$

where $p_t(x)$ is the heat kernel associated to the operator L (see Theorem 1.6.1). Then, given $\delta > 0$, $x \in B_1$ and $t \in [-1+\delta, 0)$ we can integrate the relation in time to obtain the following:

$$\begin{aligned}
w(x,t) &\le \int_{t-\delta}^{t-\frac{\delta}{2}}\int_{\mathbb{R}^n} p_{t-\zeta}(x-y)|w(y,\zeta)|\mathrm{d}y\mathrm{d}\zeta \\
&\le \int_{t-\delta}^{t-\frac{\delta}{2}}\int_{\mathbb{R}^n} C\min\{(t-\zeta)^{-\frac{n}{2s}}, (t-\zeta)|x-y|^{-n-2s}\}|w(y,\zeta)|\mathrm{d}y\mathrm{d}\zeta \\
&\le C\int_{t-\delta}^{t-\frac{\delta}{2}}\int_{\mathbb{R}^n} \frac{2}{(t-\zeta)^{\frac{n}{2s}} + (t-\zeta)^{-1}|x-y|^{n+2s}}|w(y,\zeta)|\mathrm{d}y\mathrm{d}\zeta \\
&\le C\int_{t-\delta}^{t-\frac{\delta}{2}}\int_{\mathbb{R}^n} \frac{2}{(\frac{\delta}{2})^{\frac{n}{2s}} + \delta^{-1}|x-y|^{n+2s}}|w(y,\zeta)|\mathrm{d}y\mathrm{d}\zeta \\
&\le C\int_{t-\delta}^{t-\frac{\delta}{2}}\int_{\mathbb{R}^n} \frac{1}{1+|y|^{n+2s}}|w(y,\zeta)|\mathrm{d}y\mathrm{d}\zeta,
\end{aligned}$$

and C depends on δ, and universal constants (n, s, λ and Λ). □

For the interior regularity, we will need an analogue of [94, Corollary 3.4].

Proposition 1.6.4. *Let L be an operator satisfying (1.2) and (1.3). Let $u \in L^\infty(\mathbb{R}^n \times (-1,0))$ be a viscosity solution of $u_t + Lu = f$ in $B_1 \times (-1,0)$. Assume additionally that*

$$\begin{aligned}
C_0 = &\sup_{t\in(-1,0)} \|u(\cdot,t)\|_{C^\alpha(\mathbb{R}^n)} + \sup_{x\in\mathbb{R}^n} \|u(x,\cdot)\|_{C^\beta((-1,0))} \\
&+ \sup_{t\in(-1,0)} \|f(\cdot,t)\|_{C^\alpha(B_1)} + \sup_{x\in B_1} \|f(x,\cdot)\|_{C^\beta((-1,0))} < \infty,
\end{aligned}$$

for some $\alpha, \beta \ge 0$ (with the L^∞ norm if α or β are 0).

Then, for all $\varepsilon > 0$, $u \in C_x^{\alpha+2s-\varepsilon}C_t^{\beta+1-\varepsilon}(\overline{B_{1/2}} \times [-\frac{1}{2}, 0])$, and

$$\sup_{t\in[-\frac{1}{2},0]} \|u(\cdot,t)\|_{C^{\alpha+2s-\varepsilon}(\overline{B_{1/2}})} + \sup_{x\in\overline{B_{1/2}}} \|u(x,\cdot)\|_{C^{\beta+1-\varepsilon}([-\frac{1}{2},0])} \le CC_0,$$

where C only depends on the dimension, s, ε, and the ellipticity constants.

Proof. The proof is the same as the proof of [94, Corollary 3.4], but using [180, Theorem 2.2] instead of [94, Theorem 1.3]. □

Combining the heat kernel estimates with the interior regularity result, we obtain the following bound.

Corollary 1.6.5. *Let L be an operator satisfying (1.2) and (1.3), and let p_t as introduced in Theorem 1.6.1. Then, for all $r_0 > 0$,*

$$\|\nabla p_t\|_{L^\infty(B^c_{r_0}\times(0,T))} \le C,$$

where C depends only on r_0, T, the dimension, s and the ellipticity constants.

Proof. Assume after a scaling that $r_0 = 1$. Iterating proposition 1.6.4, we obtain that

$$\sup_{t\in[-2^{-k},0]} \|p_t(\cdot,t)\|_{C^1(\overline{B_{2^{-k}}})} \le C\|p_t\|_{L^\infty(B_1\times(-1,0))},$$

for some big enough k depending only on s. After a scaling and a covering argument, for all $x \in B_1^c$ it holds

$$\begin{array}{rcll} \|\nabla p_t\|_{L^\infty(B_{\delta/2}(x)\times[t-\frac{\delta^{2s}}{2},t])} & \le & C_\delta\|p_t\|_{L^\infty(B_\delta(x)\times(t-\delta^{2s},t))}, & \text{for all} \quad t\in(\delta^{2s},T), \\ \|\nabla p_t\|_{L^\infty(B_{t^{\frac{1}{2s}}/2}(x)\times[\frac{t}{2},t])} & \le & C_0 t^{-1}\|p_t\|_{L^\infty(B_{t^{\frac{1}{2s}}}(x)\times(0,t))}, & \text{for all} \quad t\in(0,T), \end{array}$$

where we leave $\delta > 0$ to be chosen later.

Then, using Theorem 1.6.1, substituting the estimate $|p_t(x)| \le c_2 t|x|^{-n-2s}$,

$$\|\nabla p_t\|_{L^\infty(B_{\delta/2}(x)\times[t-\frac{\delta^{2s}}{2},t])} \le C_\delta c_2 t(1-\delta)^{-n-2s}, \quad \text{for all } t\in(\delta^{2s},T),$$

and

$$\|\nabla p_t\|_{L^\infty(B_{t^{\frac{1}{2s}}/2}(x)\times[\frac{t}{2},t])} \le C_0 c_2 (1-t^{\frac{1}{2s}})_+^{-n-2s}, \quad \text{for all } t\in(0,T).$$

Finally, choosing $\delta = \frac{1}{4}$, for all $x \in B_1^c$ and $t \ge 4^{-2s}$,

$$|\nabla p_t(x,t)| \le C_{1/4}c_2 t(3/4)^{-n-2s} \le C_{1/4}c_2 T(3/4)^{-n-2s},$$

and for all $x \in B_1^c$ and $t\in(0,4^{-2s})$,

$$|\nabla p_t(x,t)| \le C_0 c_2(3/4)^{-n-2s},$$

as we wanted to prove. □

We will also make use of the following estimate for the nonlocal heat equation.

Proposition 1.6.6. *Let L be an operator satisfying (1.2) and (1.3). Then, there exists $\delta > 0$ such that the following holds. If $b \in L^\infty$ is continuous and satisfies*

$$\left\{ \begin{array}{rcll} |(\partial_t + L)b| & \le & \delta\max\{|x|,1\}^{-n-2s} & \text{in} \quad \mathbb{R}^n\times(0,1) \\ b & = & b_0 & \text{on} \quad \{t=0\}, \end{array} \right.$$

where $b_0 \ge 0$, $\operatorname{supp} b_0 \subset B_1$ and $\|b_0\|_{L^1(B_1)} = 1$, the following estimate holds:

$$c_1 t|x|^{-n-2s} \le b(x,t) \le c_2 t|x|^{-n-2s} \quad \text{for all} \quad (x,t)\in B_2^c\times(0,1)$$

The constants δ, c_1 and c_2 are positive and depend only on the dimension, s and the ellipticity constants.

Proof. We will use Duhamel's formula with the fundamental solution, together with Theorem 1.6.1. Let us take $\delta = 0$ first and then we will show that the perturbation introduced by the right hand side can be absorbed by the constants.

If $|x| > 2$ and $t < 1$, $p_t(x) \asymp t|x|^{-n-2s}$. Thus, if $|x| \geq 2$, for all $y \in B_1$, $|x-y| \asymp |x|$, and then

$$\begin{aligned} b(x,t) &= \int_{\mathbb{R}^n} p_t(x-y)b_0(y)\mathrm{d}y = \int_{B_1} p_t(x-y)b_0(y)\mathrm{d}y \\ &\asymp \int_{B_1} t|x|^{-n-2s}b_0(y)\mathrm{d}y = t|x|^{-n-2s}. \end{aligned}$$

Now, if we allow a right hand side in the PDE, making $\delta > 0$, we obtain the following:

$$\left| b_R(x,t) - \int_{\mathbb{R}^n} p_t(x-y)b_0(y)\mathrm{d}y \right| \leq \delta \int_0^t \int_{\mathbb{R}^n} p_{t-\zeta}(x-y)\max\{|y|,1\}^{-n-2s}\mathrm{d}y\mathrm{d}\zeta,$$

and then we can estimate the second integral as follows. First we separate the integral in pieces, taking into account that $p_t(x-y) \lesssim \min\{t^{-\frac{n}{2s}}, t|x-y|^{-n-2s}\}$, and also that $|x| \geq 2$.

$$\begin{aligned} I_1 &:= \int_{B_1} t|x-y|^{-n-2s}\mathrm{d}y \lesssim t|x|^{-n-2s}, \\ I_2 &:= \int_{B_{t^{\frac{1}{2s}}}(x)} t^{-\frac{n}{2s}} \max\{1,|y|\}^{-n-2s}\mathrm{d}y \lesssim (t^{\frac{1}{2s}})^n t^{-\frac{n}{2s}}|x|^{-n-2s} = |x|^{-n-2s}, \\ I_3 &:= \int_{B_1(x)\setminus B_{t^{\frac{1}{2s}}}(x)} t|x-y|^{-n-2s}|y|^{-n-2s}\mathrm{d}y \lesssim t(t^{\frac{1}{2s}})^{-2s}|x|^{-n-2s} = |x|^{-n-2s}, \\ I_4 &:= \int_{B_1^c\cap B_1^c(x)} t|x-y|^{-n-2s}|y|^{-n-2s}\mathrm{d}y = t\int_{B_1^c\cap B_1^c(x)} |x-y|^{-n-2s}|y|^{-n-2s}\mathrm{d}y \\ &= 2t\int_{B_1^c\cap\{x\cdot y\leq|x|^2/2\}} |x-y|^{-n-2s}|y|^{-n-2s}\mathrm{d}y \lesssim t|x|^{-n-2s}\int_{B_1^c}|y|^{-n-2s}\mathrm{d}y \lesssim t|x|^{-n-2s}, \end{aligned}$$

where we used that $|x-y| \geq \frac{|x|}{2}$ in the half-space $\{x\cdot y \leq |x|^2/2\}$ to estimate I_4.

Putting everything together, we have

$$\int_{\mathbb{R}^n} p_t(x-y)\max\{|y|,R\}^{-n-2s}\mathrm{d}y \leq I_1+I_2+I_3+I_4 \lesssim |x|^{-n-2s}.$$

Therefore, the error term introduced by the right hand side in the PDE can be bounded by the main term:

$$\begin{aligned} \left| b_R(x,t) - \int_{\mathbb{R}^n} p_t(x-y)b_0(y)\mathrm{d}y \right| &\leq \delta \int_0^t \int_{\mathbb{R}^n} p_{t-\zeta}(x-y)\max\{|y|,1\}^{-n-2s}\mathrm{d}y\mathrm{d}\zeta \\ &\lesssim \delta t|x|^{-n-2s} \lesssim \delta \int_{\mathbb{R}^n} p_t(x-y)b_0(y)\mathrm{d}y. \end{aligned}$$

Thus, choosing δ small enough, we have $b_R(x,t) \asymp t|x|^{-n-2s}$ for $|x| \geq 2$. □

1.7 Appendix: The penalized parabolic obstacle problem

First, we need that the penalized problem has a unique solution. To do that, we first prove that there holds a comparison principle.

Lemma 1.7.1. *Let $\varepsilon > 0$, let L be a nonlocal operator satisfying (1.2) and (1.3), and let f, g, φ, ψ, u_0 and v_0 be uniformly Lipschitz and bounded, and let u and v be uniformly Lipschitz and bounded solutions of the following parabolic problems:*

$$\begin{cases} \partial_t u + Lu &= \beta_\varepsilon(u-\varphi) + f \quad \text{in} \quad \mathbb{R}^n \times (0,T) \\ u(\cdot, 0) &= u_0, \end{cases}$$

$$\begin{cases} \partial_t v + Lv &= \beta_\varepsilon(v-\psi) + g \quad \text{in} \quad \mathbb{R}^n \times (0,T) \\ v(\cdot, 0) &= v_0, \end{cases}$$

where $\beta_\varepsilon(z) = e^{-z/\varepsilon}$. Assume additionally that $u_0 \le v_0$, $\varphi \le \psi$ and $f \le g$. Then, $u \le v$ in $\mathbb{R}^n \times (0,T)$.

Proof. Assume that $\inf(v-u) < 0$, otherwise there is nothing to prove. Let $\delta > 0$ small, $M > 0$ large to be chosen later, and let $p(x) = (1+|x|)^s$. First, one can check by a direct computation that Lp is bounded. Then, the function

$$w(x,t) = v(x,t) - u(x,t) + \frac{\delta}{T-t} + \delta p(x) + \delta M$$

has an absolute minimum in $\mathbb{R}^n \times [0,T]$, and taking δ small enough, the minimum is negative. Let (x_0, t_0) be the minimum point. First, observe that, since the minimum is negative, $t_0 > 0$, because $v \ge u$ at $t = 0$. Notice also that $t_0 < T$ because $\delta(T-t)^{-1}$ tends to infinity as $t \to T$. Then, (x_0, t_0) is an interior point and then we can differentiate in t and evaluate L, which is well defined thanks to the uniform Lipschitz regularity. Therefore,

$$\begin{aligned} v_t(x_0,t_0) - u_t(x_0,t_0) + \frac{\delta}{(T-t_0)^2} &= 0 \\ Lv(x_0,t_0) - Lu(x_0,t_0) + \delta Lp(x_0) &\le 0. \end{aligned}$$

Furthermore, we can also evaluate the equations at (x_0, t_0) to obtain

$$\begin{aligned} (\partial_t + L)u(x_0,t_0) &= \beta_\varepsilon(u(x_0,t_0) - \varphi(x_0)) + f(x_0,t_0) \\ (\partial_t + L)v(x_0,t_0) &= \beta_\varepsilon(v(x_0,t_0) - \psi(x_0)) + g(x_0,t_0). \end{aligned}$$

And then combining the equations and using that β_ε is decreasing,

$$\begin{aligned} \beta(v-\varphi) - \beta(u-\varphi) &\le \beta(v-\psi) + g - \beta(u-\varphi) - f \\ &= (\partial_t + L)(v-u) \le \delta \left[Lp - \frac{1}{(T-t_0)^2} \right] \le C\delta, \end{aligned}$$

where we have omitted that all the functions are considered at the point (x_0, t_0) for ease of read. It follows that $v(x_0,t_0) - u(x_0,t_0) \ge -C'\delta$. Therefore, choosing $M > C'$, $w(x_0,t_0) > 0$, a contradiction. Therefore $v \ge u$ in $\mathbb{R}^n \times (0,T)$. □

Then, using the Perron method, one can construct a viscosity solution for the penalized problem.

Proposition 1.7.2. *For all $\varepsilon > 0$ and $\varphi \in C_c^{2,1}(\mathbb{R}^n)$, there exists a unique viscosity solution, $u^\varepsilon \in C(\mathbb{R}^n \times [0,T]) \cap L^\infty(\mathbb{R}^n \times [0,T])$, to the penalized problem*

$$\begin{cases} \partial_t u^\varepsilon + Lu^\varepsilon &= \beta_\varepsilon(u^\varepsilon - \varphi) \quad \text{in} \quad \mathbb{R}^n \times (0,T) \\ u^\varepsilon(\cdot, 0) &= \varphi + \sqrt{\varepsilon}, \end{cases}$$

where $\beta_\varepsilon(z) = e^{-z/\varepsilon}$.

Sketch of the proof. The proof follows the standard techniques in viscosity solutions, see [119] for a detailed explanation in the case of local operators.

To see existence, we construct a bounded continuous subsolution and supersolution, and then we will take the infimum of all supersolutions as our solution.

It is easy to check that $u_-(x,t) = -\|\varphi\|_{L^\infty(\mathbb{R}^n)}$ is a subsolution. Indeed,

$$u_-(\cdot, 0) \le \varphi + \sqrt{\varepsilon} \quad \text{and} \quad (\partial_t + L)u_- - \beta_\varepsilon(u_- - \varphi) = -\beta_\varepsilon(u_- - \varphi) \le 0.$$

On the other hand, $u_+(x,t) = \|\varphi\|_{L^\infty(\mathbb{R}^n)} + \sqrt{\varepsilon} + t$ is a supersolution. The initial condition is immediately fulfilled, and

$$(\partial_t + L)u_+ - \beta_\varepsilon(u_+ - \varphi) = 1 - \beta_\varepsilon(u_+ - \varphi) \ge 1 - \beta_\varepsilon(\sqrt{\varepsilon}) = 1 - e^{-1/\sqrt{\varepsilon}} > 0.$$

Then, we can apply the standard procedure for viscosity solutions and define

$$u^*(x,t) := \inf\{u(x,t) |\ u \text{ is a supersolution}\},$$

and then it can be checked that u^* is a solution in the viscosity sense. Furthermore, $u_- \le u^* \le u_+$.

By interior regularity, such solution u^* is a classical solution, and thus uniqueness follows from Lemma 1.7.1. □

Then, we prove some basic properties of solutions to this problem. The following lemma is analogous to the first part of [42, Lemma 3.3] for our case, and the proof is very similar.

Lemma 1.7.3. *Let L be an operator satisfying (1.2) and (1.3), let $\varphi \in C_c^{2,1}(\mathbb{R}^n)$ and let u^ε be the solution of (1.5).*

Then,

$$\beta_\varepsilon(u^\varepsilon - \varphi) \le \max\{1, \|L\varphi\|_{L^\infty(\mathbb{R}^n)}\}.$$

In particular,

$$u^\varepsilon - \varphi \ge -\varepsilon \ln^+ \|L\varphi\|_{L^\infty(\mathbb{R}^n)}.$$

Proof. If $u^\varepsilon \ge \varphi$ everywhere, then $\beta_\varepsilon \le 1$ and there is nothing to prove. Assume then otherwise, i.e., $\inf_{\mathbb{R}^n \times [0,T]} (u^\varepsilon - \varphi) < 0$.

Then, since $u^\varepsilon \in L^\infty(\mathbb{R}^n \times (0,T))$, if $p(x) = (1 + |x|)^s$ as in Lemma 1.7.1, for any $\delta > 0$ the function

$$w = u^\varepsilon - \varphi + \frac{\delta}{T - t} + \delta p$$

has a minimum point $(x_\varepsilon^\delta, t_\varepsilon^\delta) \in \mathbb{R}^n \times [0,T]$. Furthermore, if δ is small enough, $w(x_\varepsilon^\delta, t_\varepsilon^\delta) < 0$, and it follows that $t_\varepsilon^\delta \in (0,T)$. Hence, since the point is interior and u^ε is smooth, then $\partial_t w(x_\varepsilon^\delta, t_\varepsilon^\delta) = 0$ and $Lw(x_\varepsilon^\delta, t_\varepsilon^\delta) \le 0$, which combined with the penalized equation (1.5) yields

$$\beta_\varepsilon(u^\varepsilon - \varphi)(x_\varepsilon^\delta, t_\varepsilon^\delta) \le L\varphi(x_\varepsilon^\delta) - \frac{\delta}{(T - t_\varepsilon^\delta)^2} - \delta Lp(x_\varepsilon^\delta) \le \|L\varphi\|_{L^\infty(\mathbb{R}^n)} + C\delta.$$

Finally, since β_ε is decreasing and $(u^\varepsilon - \varphi)(x_\varepsilon^\delta, t_\varepsilon^\delta) \to \inf_{\mathbb{R}^n \times [0,T]} (u^\varepsilon - \varphi)$ as $\delta \to 0$, we obtain that

$$\sup_{\mathbb{R}^n \times [0,T]} \beta_\varepsilon(u^\varepsilon - \varphi) \le \|L\varphi\|_{L^\infty(\mathbb{R}^n)},$$

as wanted. □

We can also prove an upper bound for u^ε.

Lemma 1.7.4. *Let L be an operator satisfying (1.2) and (1.3), let $\varphi \in C_c^{2,1}(\mathbb{R}^n)$ and let u^ε be the solution of (1.5).*

Then,

$$u^\varepsilon(\cdot, t) - \varphi \le \sqrt{\varepsilon} + 2t \max\{1, \|L\varphi\|_{L^\infty(\mathbb{R}^n)}\}.$$

Proof. First, let us compute

$$(\partial_t + L)(u^\varepsilon - \varphi) = \beta_\varepsilon(u^\varepsilon - \varphi) - L\varphi \le \max\{1, \|L\varphi\|_{L^\infty(\mathbb{R}^n)}\} + \|L\varphi\|_{L^\infty(\mathbb{R}^n)},$$

where we used Lemma 1.7.3 to estimate β_ε.

Therefore, if we define

$$w(x,t) = u^\varepsilon(x,t) - \varphi(x) - \sqrt{\varepsilon} - 2t \max\{1, \|L\varphi\|_{L^\infty(\mathbb{R}^n)}\},$$

we get that $w(\cdot, 0) \equiv 0$ and that w is a subsolution, $(\partial_t + L)w \le 0$, by construction, and then the comparison principle for classical solutions of the nonlocal parabolic equation yields the result. □

We also need to see that we can differentiate the penalized problem.

Lemma 1.7.5. *Let L be an operator satisfying (1.2) and (1.3), let $\varphi \in C_c^{2,1}(\mathbb{R}^n)$ and let u^ε be the solution to the penalized problem (1.5). Then, given any unit vector $\nu \in \mathbb{R}^n \times \mathbb{R}$,*

$$\begin{aligned}
(\partial_t + L)u_\nu^\varepsilon &= \beta_\varepsilon'(u^\varepsilon - \varphi)(u_\nu^\varepsilon - \varphi_\nu), \\
(\partial_t + L)u_{\nu\nu}^\varepsilon &= \beta_\varepsilon'(u^\varepsilon - \varphi)(u_{\nu\nu}^\varepsilon - \varphi_{\nu\nu}) + \beta_\varepsilon''(u^\varepsilon - \varphi)(u_\nu^\varepsilon - \varphi_\nu)^2, \\
u_t^\varepsilon(\cdot, 0) &= -L\varphi + e^{-1/\sqrt{\varepsilon}}, \\
u_{tt}^\varepsilon(\cdot, 0) &= L^2\varphi - \frac{1}{\varepsilon} e^{-1/\sqrt{\varepsilon}}(e^{-1/\sqrt{\varepsilon}} - L\varphi),
\end{aligned}$$

where the two last expressions must be understood in the sense of the uniform limit as $t \to 0^+$.

Proof. We will iterate Proposition 1.6.4. First, $u^\varepsilon \in L^\infty(\mathbb{R}^n \times (0,T))$ by Lemmas 1.7.3 and 1.7.4. Then, observe that $\beta_\varepsilon(u^\varepsilon - \varphi) \in L^\infty$ as well.

Let $W = W_x \times [t_1, t_2]$ be a compact cylinder in $\mathbb{R}^n \times (0,T)$. Then, by Proposition 1.6.4 and a covering argument, $\|u^\varepsilon\|_{C_x^{2s-\delta}C_t^{1-\delta}(W)} \leq C$ for a small $\delta > 0$ to be chosen later. Since W is arbitrary,

$$u^\varepsilon \in C_x^{2s-\delta}C_t^{1-\delta}(\mathbb{R}^n \times (0,T)),$$

and, since the previous estimates where invariant with respect to translations in x,

$$\|u^\varepsilon\|_{C_x^{2s-\delta}C_t^{1-\delta}(\mathbb{R}^n\times[t_1,t_2])} \leq C(t_1,t_2),$$

for any $0 < t_1 < t_2 < T$.

Now, repeating the same argument k times we obtain that

$$\|u^\varepsilon\|_{C_x^{3,2s-\delta}C_t^{k(1-\delta)}(\mathbb{R}^n\times[t_1,t_2])} \leq C(t_1,t_2),$$

provided that k is large enough. The cap in the x regularity comes from the fact that $\varphi \in C_c^{2,1}$ and then $\beta_\varepsilon(u^\varepsilon - \varphi)$ cannot attain further regularity in x.

In particular, $u^\varepsilon \in C^3(\mathbb{R}^n \times (0,T))$, it is a classical solution, and then u^ε_ν and $u^\varepsilon_{\nu\nu}$ are at least C^1 in $\mathbb{R}^n \times (0,T)$, and they are also bounded for each $t \in (0,T)$, and therefore they are also classical solutions of their respective equations.

For the initial conditions, we recover the expression of u^ε from Duhamel's formula,

$$u^\varepsilon = p_t * \varphi + \sqrt{\varepsilon} + \int_0^t p_\tau * \Big(\beta(u^\varepsilon(\cdot, t-\tau) - \varphi)\Big)\mathrm{d}\tau,$$

and then differentiate it with respect to t to get

$$u^\varepsilon_t = \partial_t p_t * \varphi + p_t * \beta|_{t=0} + \int_0^t p_\tau * \Big(\beta'(u^\varepsilon(\cdot, t-\tau) - \varphi)u^\varepsilon_t(\cdot, t-\tau)\Big)\mathrm{d}\tau.$$

Then, notice that $\partial_t p_t = -Lp_t$ because p_t is a solution, and it follows that $\partial_t p_t * \varphi = p_t * (-L\varphi)$. Furthermore, $\beta(u^\varepsilon - \varphi) \equiv e^{-1/\sqrt{\varepsilon}}$ at $t = 0$, so putting everything together,

$$u^\varepsilon_t = -p_t * (L\varphi) + e^{-1/\sqrt{\varepsilon}} - \frac{1}{\varepsilon}\int_0^t p_\tau * \Big(\beta(u^\varepsilon(\cdot, t-\tau) - \varphi)u^\varepsilon_t(\cdot, t-\tau)\Big)\mathrm{d}\tau. \tag{1.6}$$

Since p_t is an approximation to the identity (see Corollary 1.6.2) and β is bounded by Lemma 1.7.3, taking the L^∞ norm we can conclude that

$$\|u^\varepsilon_t(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq C_1 + C_2\int_0^t \|u^\varepsilon_t(\cdot, \tau)\|_{L^\infty(\mathbb{R}^n)}\mathrm{d}\tau,$$

which implies by the Gronwall inequality that $u^\varepsilon_t \in L^\infty(\mathbb{R}^n \times (0,t))$.

Then, again by (1.6), since β and u^ε_t are bounded, $L\varphi$ is uniformly C^2 and p_t is an approximation to the identity, it follows that $u^\varepsilon_t \to -L\varphi + e^{-1/\sqrt{\varepsilon}}$ uniformly as $t \to 0^+$.

For the last identity, we first differentiate (1.6) with respect to time to obtain

$$u^\varepsilon_{tt} = -\partial_t p_t * (L\varphi) - \frac{1}{\varepsilon}p_t * (\beta u^\varepsilon_t)|_{t=0} - \frac{1}{\varepsilon}\int_0^t p_\tau * (\beta'(u^\varepsilon_t)^2 + \beta u^\varepsilon_{tt})(\cdot, t-\tau)\mathrm{d}\tau. \tag{1.7}$$

Now, by the same arguments used to simplify (1.6),

$$u_{tt}^{\varepsilon} = p_t * \left(L^2\varphi - \frac{1}{\varepsilon}e^{-1/\sqrt{\varepsilon}}(-L\varphi + e^{-1/\sqrt{\varepsilon}})\right) - \frac{1}{\varepsilon}\int_0^t p_\tau * (\beta'(u_t^{\varepsilon})^2 + \beta u_{tt}^{\varepsilon})(\cdot, t-\tau)\mathrm{d}\tau,$$

and then using the boundedness of u_t^{ε} and a Gronwall inequality, analogously to what we did with u_t^{ε},

$$u_{tt}^{\varepsilon} \to L^2\varphi - \frac{1}{\varepsilon}e^{-1/\sqrt{\varepsilon}}(-L\varphi + e^{-1/\sqrt{\varepsilon}}),$$

uniformly as $t \to 0^+$. □

Finally, we prove that the solutions to the penalized problem converge to the solution to the obstacle problem.

Proof of Lemma 1.2.2. Let $\varepsilon \in (0,1)$.

Step 1. First, recall the L^∞ estimates for $u^\varepsilon - \varphi$. From Lemmas 1.7.3 and 1.7.4,

$$-\varepsilon \ln^+ \|L\varphi\|_{L^\infty(\mathbb{R}^n)} \le u^\varepsilon - \varphi \le \sqrt{\varepsilon} + 2t\max\{1, \|L\varphi\|_{L^\infty(\mathbb{R}^n)}\}.$$

Now we use interior estimates and Arzelà-Ascoli to show that $u^\varepsilon \to u^0$ locally uniformly along a subsequence.

Let $W \subset\subset \mathbb{R}^n \times (0,T)$. Then, we can apply a version of [94, Theorem 1.3] to obtain

$$\|u^\varepsilon\|_{C_t^{1-\delta}(W)} + \|u^\varepsilon\|_{C_x^{2s(1-\delta)}(W)} \le C\Big(\|u^\varepsilon\|_{L^\infty(\mathbb{R}^n\times(0,T))} + \|\beta_\varepsilon(u^\varepsilon - \varphi)\|_{L^\infty(\mathbb{R}^n\times(0,T))}\Big) \le C,$$

with C only depending on W, $\|L\varphi\|_{L^\infty(\mathbb{R}^n)}$, $\delta > 0$, the dimension, s, and the ellipticity constants, because of the previous L^∞ estimates on u^ε and β_ε.

Hence, choosing a suitable small δ, by the compact inclusion of Hölder spaces and Arzelà-Ascoli, $u^{\varepsilon_k} \to u^0$ uniformly in W for some subsequence $\varepsilon_k \to 0$.

Now, consider a sequence of compact sets $W_0 \subset W_1 \subset \ldots$ such that their union is $\mathbb{R}^n \times (0,T)$ and repeat the same reasoning above. By a standard diagonalization argument, we can construct a sequence ε_k such that $u^{\varepsilon_k} \to u^0$ locally uniformly in $\mathbb{R}^n \times (0,T)$.

Step 2. Putting it together, we want to prove that, for all $\kappa > 0$, $u^{\varepsilon_k} \to u^0$ also in the $L^\infty([0, T-\kappa] \to L_s^1)$ norm. To do it, let $\tau > 0$ to be chosen later. Then, we distinguish two cases. If $t < \tau$,

$$\begin{aligned}\|u^{\varepsilon_k}(\cdot,t) - u^0(\cdot,t)\|_{L_s^1} &\le \|u^{\varepsilon_k}(\cdot,t) - \varphi\|_{L_s^1} + \|\varphi - u^0(\cdot,t)\|_{L_s^1} \\ &\le 2\sup_{m\ge k}\|u^{\varepsilon_m} - \varphi\|_{L_s^1} \le 2C\sup_{m\ge k}\|u^{\varepsilon_m} - \varphi\|_{L^\infty(\mathbb{R}^n)} \\ &< 2C\Big(\sqrt{\varepsilon_k} + 2\tau\max\{1, \|L\varphi\|_{L^\infty(\mathbb{R}^n)}\}\Big).\end{aligned}$$

On the other hand, if $t \ge \tau$ we use the locally uniform convergence of the sequence. Let $R > 0$. Then, for all $t \in [\tau, T-\kappa]$,

$$\begin{aligned}\|u^{\varepsilon_k}(\cdot,t) - u^0(\cdot,t)\|_{L_s^1} &\lesssim \|u^{\varepsilon_k}(\cdot,t) - u^0(\cdot,t)\|_{L^\infty(B_R)} + R^{-2s}\|u^{\varepsilon_k}(\cdot,t) - u^0(\cdot,t)\|_{L^\infty(B_R^c)} \\ &\lesssim \|u^{\varepsilon_k}(\cdot,t) - u^0(\cdot,t)\|_{L^\infty(B_R)} + 2R^{-2s}\sup_{m\ge k}\|u^{\varepsilon_m}(\cdot,t)\|_{L^\infty(B_R^c)} \\ &\lesssim \|u^{\varepsilon_k}(\cdot,t) - u^0(\cdot,t)\|_{L^\infty(B_R)} + R^{-2s},\end{aligned}$$

and then the second term tends to zero as $R \to \infty$ and then the first term tends to zero as k goes to infinity by the local uniform convergence.

Therefore, choosing first τ small, then R big and then k big, $\|u^{\varepsilon_k}(\cdot,t) - u^0(\cdot,t)\|_{L^1_s}$ can be made arbitrarily small, as we wanted to see.

Step 3. Then we prove that u^0 is the solution of (1.1).

First, from the lower bound $u^{\varepsilon_k} \geq \varphi - \varepsilon_k \ln^+ \|L\varphi\|_{L^\infty(\mathbb{R}^n)}$, taking the limit $\varepsilon_k \to 0$ it becomes clear that $u^0 \geq \varphi$. Then $(\partial_t + L)u^{\varepsilon_k} = \beta_{\varepsilon_k}(u^{\varepsilon_k} - \varphi) \geq 0$, and the uniform limit of viscosity supersolutions is also a supersolution (with the extra convergence assumption of Step 2), by [55, Theorem 5.3].

Hence, we only need to check that $(\partial_t + L)u^0 = 0$ in the set $\{u^0(x,t) > \varphi(x)\}$ in the viscosity sense. Again by [55, Theorem 5.3], it suffices to check the following.

Consider a compact set $E \subset \{u^0(x,t) > \varphi(x)\}$. By the uniform convergence of u^{ε_k} to u_0, there exist $\mu > 0$ and k_0 such that for all $k \geq k_0$, $u^{\varepsilon_k}(x,t) > \varphi(x) + \mu$, for all $(x,t) \in E$. Hence,

$$(\partial_t + L)u^{\varepsilon_k}(x,t) = \beta_{\varepsilon_k}(u^{\varepsilon_k} - \varphi)(x,t) \in (0, e^{-\mu/\varepsilon_k}),$$

and the limit of the right hand side is zero when $\varepsilon_k \to 0$.

Finally, from the L^∞ estimates in Lemmas 1.7.3 and 1.7.4, it follows the concordance of the initial conditions, $u^0(\cdot,0) = \varphi$, and the continuity of u^0 as $t \to 0^+$.

Step 4. Using the uniqueness of the solution we can eliminate the need to consider subsequences. Indeed, for any $\varepsilon_n \downarrow 0$, we can repeat Steps 2 and 3 to obtain a subsequence $u^{\varepsilon_{n_j}}$ that converges locally uniformly to the solution of (1.1). Therefore, $u^\varepsilon \to u^0$ locally uniformly as well. □

Chapter 2

Generic regularity of free boundaries for the thin obstacle problem

The free boundary for the Signorini problem in $\mathbb{R}^{n+1}$ is smooth outside of a degenerate set, which can have the same dimension $(n-1)$ as the free boundary itself.

In [96] it was shown that *generically*, the set where the free boundary is not smooth is at most $(n-2)$-dimensional. Our main result establishes that, in fact, the degenerate set has zero $\mathcal{H}^{n-3-\alpha_0}$ measure for a generic solution. As a by-product, we obtain that, for $n+1 \leq 4$, the whole free boundary is generically smooth. This solves the analogue of a conjecture of Schaeffer in $\mathbb{R}^3$ and $\mathbb{R}^4$ for the thin obstacle problem.

2.1 Introduction

The Signorini problem (also known as the thin or boundary obstacle problem) is a classical free boundary problem that was originally studied by Antonio Signorini in connection with linear elasticity [183, 184, 127]. The same equations appear in a variety of settings such as Biology, Fluid Mechanics, and Finance, and they have received a lot of interest from different areas [81, 151, 66, 165, 91].

The thin obstacle problem is equivalent to the obstacle problem for the half-Laplacian $(-\Delta)^{1/2}$, and has been extensively studied by the mathematical community in the last two decades; see [35, 7, 49, 11, 110, 160, 133, 72, 69, 107, 134, 65, 191, 93, 108], and the references therein. In particular, the study of the Signorini problem is a crucial ingredient to understand the free boundary in the *thick* obstacle problem [101, 102, 175, 174].

Obstacle problems belong to a wide class of problems known as *free boundary problems*, where one of the unknowns is the contact set, and more precisely, its boundary, the free boundary. There are explicit constructions [177] for the classical obstacle problem that give rise to free boundaries having a set of singular points of the same dimension as the whole free boundary. Still, singular points are expected to be *infrequent*: Schaeffer conjectured in 1974 ([176]) that, for a generic boundary datum, the free boundary is regular. The conjecture was proved to hold true in the plane $\mathbb{R}^2$ by Monneau in [154], and much more recently in a breakthrough work, [102], Figalli, Ros-Oton, and Serra showed that it also holds in $\mathbb{R}^3$ and $\mathbb{R}^4$.

Given the parallels between the classical obstacle problem and the thin obstacle problem, it is natural to extend the conjecture of Schaeffer to the setting of the latter:

Conjecture 2.1.1. *Generically, the free boundary in the Signorini problem is smooth.*

Also for the thin obstacle problem, there are examples of particular solutions having non-regular points of the same dimension as the whole free boundary (see e.g. [110, 96]). The validity of the previous conjecture would imply that such solutions are *rare*.

Conjecture 2.1.1 was recently proved in $\mathbb{R}^2$ by the first author and Ros-Oton in [96] (with operators $\mathrm{div}(|x_{n+1}|^a\nabla\cdot)$ for $a \in (-1,1)$). In this work, we will extend its validity to the physical dimension $\mathbb{R}^3$, and $\mathbb{R}^4$. Moreover, we will also provide dimensional estimates for the size of the set of degenerate points for dimensions $n+1 \geq 5$.

2.1.1 The Signorini problem and the free boundary

The Signorini problem with zero obstacle (originally introduced as the Laplace equation with ambiguous boundary conditions) can be written as

$$\begin{cases} \Delta u &= 0 \quad \text{in} \quad B_1^+ \\ \min\{u, -\partial_{x_{n+1}} u\} &= 0 \quad \text{on} \quad B_1 \cap \{x_{n+1} = 0\}. \end{cases} \tag{2.1}$$

Alternatively, we study the problem posed in the whole ball $B_1 \subset \mathbb{R}^{n+1}$ (extending by even symmetry) as

$$\begin{cases} \Delta u &= 0 & \text{in} \quad B_1 \setminus \{x_{n+1} = 0\} \\ \min\{u, -\Delta u\} &= 0 & \text{on} \quad B_1 \cap \{x_{n+1} = 0\} \\ u(x', x_{n+1}) &= u(x', -x_{n+1}) & \text{in} \quad B_1, \end{cases} \tag{2.2}$$

where now Δu needs to be understood in the sense of distributions. For the Signorini problem, solutions are always $C^{1,1/2}$ (on each side in (2.2), see [7]).

Like the obstacle problem, the Signorini problem is a free boundary problem. That is, one of the unknowns of the problem is the contact set

$$\Lambda(u) := \{x' \in \mathbb{R}^n : u(x', 0) = 0\} \times \{0\},$$

and in particular, its boundary (in the relative topology on the thin space), the *free boundary*

$$\Gamma(u) := \partial\{x' \in \mathbb{R}^n : u(x', 0) = 0\} \times \{0\}.$$

The free boundary has been mainly studied so far by means of blow-up methods. Namely, assume that u is a solution to (2.2) with $0 \in \Gamma(u)$, and define the blow-up sequence

$$u_r(x) := \frac{u(rx)}{\|u\|_{L^2(\partial B_r)}}. \tag{2.3}$$

It can be shown that, up to a subsequence $r_k \downarrow 0$, u_r converges (locally uniformly) to a global κ-homogeneous solution u_0. The value κ is what we call the order or frequency of the free boundary point.

The free boundary is divided into *regular points*, $\mathrm{Reg}(u)$ (with homogeneity $\kappa = 3/2$), and *degenerate points*, $\mathrm{Deg}(u)$ (with homogeneity $\kappa \geq 2$), [11]:

$$\Gamma(u) = \mathrm{Reg}(u) \cup \mathrm{Deg}(u).$$

Moreover, *for almost every solution*, the dimension of the set of degenerate points is at most $n-2$, so they are *rare* [96]. We refer to [160, 91] for more details about the structure of the free boundary, and the thin obstacle problem in general.

2.1.2 Main results

We prove that *generically*, the set of degenerate points is empty in dimensions $n+1=3$ and $n+1=4$. More precisely, we consider monotone families of solutions as follows.

Let $u : B_1 \times [-1,1] \to \mathbb{R}$ be such that $u(\cdot,t)$ solves (2.2) for each $t \in [-1,1]$ and

$$\begin{cases} u(\cdot,t') - u(\cdot,t) & \geq & 0 & \text{in} & \overline{B_1} \\ u(\cdot,t') - u(\cdot,t) & \geq & t'-t & \text{on} & \partial B_1 \cap \{|x_{n+1}| \geq \frac{1}{2}\} \\ \|u(\cdot,t)\|_{C^{0,1}(B_1)} & \leq & 1, & & \end{cases} \tag{2.4}$$

for all $-1 \leq t < t' \leq 1$. As there is no room for confusion, we will say that $u : B_1 \times [-1,1] \to \mathbb{R}$ solves (2.2) if $u(\cdot,t)$ solves it for all $t \in [-1,1]$. Our main result is the following:

Theorem 2.1.2. *Let $u : B_1 \times [-1,1] \to \mathbb{R}$ be a solution to (2.2)-(2.4). Then, for almost every $t \in [-1,1]$,*

(a) If $n \leq 3$, $\mathrm{Deg}(u(\cdot,t)) = \emptyset$.

(b) If $n \geq 4$, $\dim_{\mathcal{H}}(\mathrm{Deg}(u(\cdot,t))) \leq n-3-\alpha_\circ$, for some $\alpha_\circ > 0$ depending only on n.

Here, $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension of a set; see for example [150, Chapter 4]. We actually prove stronger results for several subsets of the free boundary, see Proposition 2.6.1. See also subsection 2.2.5 for a sketch of the proof of Theorem 2.1.2.

As a consequence of our main result we obtain that, generically, free boundaries are smooth in $\mathbb{R}^3$ and $\mathbb{R}^4$, thus showing that the analogue of Schaeffer's conjecture for the thin obstacle problem holds true in these dimensions.

Corollary 2.1.3. *Conjecture 2.1.1 holds in $\mathbb{R}^3$ and $\mathbb{R}^4$.*

We recall that this was only known in $\mathbb{R}^2$, [96].

Remark 2.1.4. The notion of genericity needs to be understood in the context of the theory of prevalence, [118] (see also [158]). In this language, we will prove that the set of solutions satisfying that the free boundary has an empty degenerate set is *prevalent* within the set of solutions in $\mathbb{R}^3$ and $\mathbb{R}^4$ (say, given by C^0 or L^∞ boundary data). Alternatively, we show that the set of solutions whose degenerate set is non-empty is *shy*. In particular, this means that for almost every boundary data (see [158, Definition 3.1]) the corresponding solution has a smooth free boundary (by [133, 72]).

Remark 2.1.5. The result in Corollary 2.1.3 is in correspondence with the results in the *thick* case in [102], in $\mathbb{R}^3$ and $\mathbb{R}^4$ as well. Part of the appeal of the present manuscript is that, due to the nature of the problem, the methods developed in [102] become much simpler in the context of the Signorini problem (once combined with [65, 96, 93, 175]), allowing us to obtain an equally strong result with far fewer technical details. Indeed, in our case, the free boundary is a set of co-dimension 2 (instead of co-dimension 1), making it a set of zero harmonic capacity. This implies, in particular, that the second-order expansion around singular points is harmonic (see Propositions 2.2.8 and 2.2.10). Conversely, in the *thick case*, the second-order term in the expansion around singular points can have different behaviors, one of them being, precisely, a solution to a thin obstacle problem, that also needs to account for the curvature of the contact set around the point, and has a different thin space at each point. Roughly speaking, the role played by $u - p$ in the thick case (where p is the first order expansion around a free boundary point, that depends on the point), is now played by u (which is the same at all points, thus allowing for a simpler analysis).

In the same way, this also means that the dimension in which Conjecture 2.1.1 holds cannot be improved only using the approach in [102]. (More specifically, completely new ideas are needed to improve the generic size of the set $\Gamma_2^{\mathrm{a}}(u)$; see subsections 2.2.2 and 2.2.5.)

Remark 2.1.6. In this work, we deal with the Signorini problem with zero obstacle, (2.1) or (2.2) (as in [65, 93, 175]), which is a model case including the problem with an analytic obstacle.

Indeed, given a function $\varphi : B_1' \subset \mathbb{R}^n \to \mathbb{R}$ where B_1' denotes the unit ball in $\mathbb{R}^n$, the Signorini problem with obstacle φ is

$$\begin{cases} \Delta u &= 0 \quad \text{in} \quad B_1 \cap \{x_{n+1} > 0\} \\ \min\{u(x',0) - \varphi(x'), -\partial_{x_{n+1}} u(x',0)\} &= 0 \quad \text{for} \quad x' \in B_1'. \end{cases}$$

When φ is analytic, it can be extended to a harmonic function in $B_1 \subset \mathbb{R}^{n+1}$ (i.e., with $\tilde{\varphi}(x',0) = \varphi(x')$ for all $x' \in B_1$), even in the last coordinate, so that $v := u - \tilde{\varphi}$ is a solution to the Signorini problem with zero obstacle, (2.2). That is, our result also applies to analytic obstacles.

Remark 2.1.7. Apart from the aforementioned works, [102, 96], the recent preprints [99] and [61, 62] obtain similar results using related techniques in the context of the Alt-Caffarelli and Alt-Phillips functionals, and minimal surfaces, respectively.

2.1.3 Plan of the paper

This paper is organized as follows.

In Section 2.2 we introduce some technical tools, such as the frequency formula, and some preliminary results. We also sketch the strategy of the proof of Theorem 2.1.2 at the end of the section. Then, the goal of Section 2.3 is to recover the known dimensional bounds for $\mathrm{Deg}(u)$ and one of its subsets, that we denote $\Gamma_*(u)$ (see (2.5)), but for a monotone family of solutions (instead of a single solution). In Section 2.4 we study the points of order 2, separating them into *ordinary quadratic points*, for which we show an improved cleaning; and *anomalous quadratic points*, for which we perform a further dimension reduction; and in Section 2.5 we study the cubic points. Finally, in Section 2.6 we combine our results to compute the final dimensional estimates.

2.2 Preliminaries

In this section we recall some background results and we develop some technical tools that will be useful later. We start with the following Liouville-type result.

Lemma 2.2.1. *Let $u : \mathbb{R}^{n+1} \to \mathbb{R}$ be a κ-homogeneous solution to (2.2). Then,*

(a) If $u \geq 0$, then $u \equiv 0$.

(b) If $u \leq 0$ and $\kappa > 1$, then $u \equiv 0$.

(c) If $\partial_e u \geq 0$ for some direction e and $\kappa \geq 2$, then u is invariant in the direction e.

Proof. (a) Suppose u is not identically zero. Then, by the Hopf lemma $\partial_{n+1}u(0,0^+) > 0$, which together with u being even in the x_{n+1} direction contradicts the fact that u is superharmonic across the thin space $\{x_{n+1} = 0\}$.

(b) Suppose u is not identically zero. Then, by the Hopf lemma $\partial_{n+1}u(0,0^+) < 0$. On the other hand, $\nabla u(0) = 0$ because the homogeneity of u is $\kappa > 1$. A contradiction.

(c) First, $\partial_e u(0) = 0$ because $\kappa \geq 2$. Assume by contradiction that $\partial_e u > 0$ in $\{x_{n+1} > 0\}$, and thus by the Hopf lemma $\partial_{n+1}\partial_e u(0,0^+) > 0$. Therefore, $D^2u(0) \not\equiv 0$, which in turn implies $\kappa = 2$, and it follows by [11, Theorem 3] that $u(x) = \sum_{i=1}^n a_i(x_i^2 - x_{n+1}^2)$ with $a_i \geq 0$, after a change of coordinates if necessary. Hence, $\partial_e u$ is linear and since $\partial_e u \geq 0$, we get $\partial_e u \equiv 0$, a contradiction. $\square$

We continue with a Hopf-type estimate to quantify the monotonicity of the families of solutions near the thin space.

Lemma 2.2.2. *Let $u : B_1 \times [-1,1] \to \mathbb{R}$ be a solution to (2.2)-(2.4). Then, for all $t \geq 0$,*

$$h_t(x) := u(x,t) - u(x,0) \geq ct|x_{n+1}| \text{ in } B_{1/2},$$

for some $c > 0$ depending only on n.

Proof. By (2.4), $h_t \geq 0$ in B_1, and $h_t \geq t$ on $\partial B_1 \cap \{|x_{n+1}| \geq \frac{1}{2}\}$. Let φ be such that $\varphi = 1$ on $\partial B_1 \cap \{|x_{n+1}| \geq \frac{1}{2}\}$, $\varphi = 0$ on $\partial B_1 \cap \{|x_{n+1}| < \frac{1}{2}\}$ and $\{x_{n+1} = 0\}$, and $\Delta\varphi = 0$ in $B_1 \cap \{x_{n+1} \neq 0\}$. Then, on the one hand, thanks to the Hopf Lemma we have that $\varphi \geq c|x_{n+1}|$ in $B_{1/2}$ for some c depending only on n; and on the other hand, by the maximum principle, $\varphi \leq \frac{h_t}{t}$ in B_1. $\square$

Given $u : B_1 \times [-1,1] \to \mathbb{R}$ a family of solutions of (2.2)-(2.4), we define the free boundary

$$\Gamma(u(\cdot,t)) = \partial\{x' \in \mathbb{R}^n : u((x',0),t) = 0\} \times \{0\},$$

and we denote

$$\mathbf{\Gamma} := \bigcup_{t\in[-1,1]} \Gamma(u(\cdot,t)).$$

Analogously, we will denote by Reg and Deg the union of all regular and degenerate points for a family of solutions. For our setting, it is convenient to define the following map:

Proposition 2.2.3 ([96, Corollary 2.7])**.** *Let $u : B_1 \times [-1,1] \to \mathbb{R}$ be a solution to (2.2)-(2.4). Then, the mapping $\tau : \mathbf{\Gamma} \to [-1,1]$ defined as $\tau(x_0) = t_0$ such that $x_0 \in \Gamma(u(\cdot,t_0))$ is well defined and continuous. Moreover, for any $\varepsilon > 0$, the map*

$$\mathbf{\Gamma} \cap B_{1-\varepsilon} \ni x_0 \mapsto u(x_0 + \cdot, \tau(x_0))$$

is continuous in the C^0 norm.

2.2.1 The frequency formula

Here, we recall and prove some facts about Almgren's frequency function.

Given $w \in H^1$loc, we define

$$\phi(r, w) := \frac{D(r, w)}{H(r, w)},$$

where

$$D(r, w) := r^{1-n} \int_{B_r} |\nabla w|^2 \quad \text{and} \quad H(r, w) := r^{-n} \int_{\partial B_r} w^2.$$

We recall that the frequency function ϕ is nondecreasing in r for solutions of (2.2):

Lemma 2.2.4 ([11, Lemma 1])**.** *Let u be a solution to (2.2). Then, the function $r \mapsto \phi(r, u)$ is nondecreasing. Moreover, $\phi(r, u)$ is constant with respect to r, $\phi(r, u) \equiv \lambda$, if and only if u is λ-homogeneous.*

This justifies that the frequency of a point x_0, $\phi(0^+, u(x_0 + \cdot))$, is always well defined; and hence, we can stratify the free boundary according to the frequency κ as follows (see Proposition 2.2.3):

$$\Gamma_\kappa(u(\cdot, t)) := \{x_0 \in \Gamma(u(\cdot, t)) : \phi(0^+, u(x_0 + \cdot, t)) = \kappa\}, \quad \mathbf{\Gamma}_\kappa := \bigcup_{t \in [-1,1]} \Gamma_\kappa(u(\cdot, t)),$$

and we also introduce the sets

$$\begin{aligned} \Gamma_{\geq\kappa}(u(\cdot, t)) &:= \bigcup_{\nu \geq \kappa} \Gamma_\nu(u(\cdot, t)), \quad \mathbf{\Gamma}_{\geq\kappa} := \bigcup_{\nu \geq \kappa} \mathbf{\Gamma}_\nu, \\ \Gamma_*(u(\cdot, t)) &:= \bigcup_{\nu \in \mathbb{R} \setminus S} \Gamma_\nu(u(\cdot, t)), \quad \mathbf{\Gamma}_* := \bigcup_{\nu \in \mathbb{R} \setminus S} \mathbf{\Gamma}_\nu, \end{aligned} \tag{2.5}$$

where $S = \{1, \frac{3}{2}, 2, 3, \frac{7}{2}, 4, \dots\} = \mathbb{N} \cup \{2\mathbb{N} - \frac{1}{2}\}$ is the set of possible homogeneities of the solutions of Signorini in dimension $n + 1 = 2$.

Observe that the frequency function can act as a proxy for the growth rate of a function:

Lemma 2.2.5. *Let $u : B_1 \to \mathbb{R}$ be a solution to (2.2). Suppose that for $0 < r < R < 1$ we have $\underline{\lambda} \leq \phi(r, u) \leq \phi(R, u) \leq \overline{\lambda}$. Then,*

$$\left(\frac{R}{r}\right)^{2\underline{\lambda}} \leq \frac{H(R, u)}{H(r, u)} \leq \left(\frac{R}{r}\right)^{2\overline{\lambda}}.$$

Proof. Let $u_r := u(r\cdot)$. Then, $H(r, u) = \int_{\partial B_1} u_r^2$, and integrating by parts,

$$H'(r, u) = \frac{2}{r} \int_{\partial B_1} u_r \partial_\nu u_r = \frac{2}{r} \left(\int_{B_1} |\nabla u_r|^2 + \int_{B_1} u_r \Delta u_r \right) = \frac{2}{r} D(u, r),$$

because $u_r \Delta u_r = 0$ for solutions of (2.2), and hence

$$\frac{H'(r, u)}{H(r, u)} = \frac{2}{r} \phi(r, u).$$

Then, integrating from r to R (and since ϕ is monotone nondecreasing, see Lemma 2.2.4),

$$2\underline{\lambda} \ln(R/r) \leq \ln\left(\frac{H(R, u)}{H(r, u)}\right) \leq 2\overline{\lambda} \ln(R/r),$$

and the conclusion follows. $\square$

Finally, once the frequency is properly defined, we may recall two results that will be used later. The first one is a strong comparison principle, from which we copy the proof for the convenience of the reader.

Lemma 2.2.6 ([102, Lemma A.4])**.** *Let u, v be two solutions of (2.2) satisfying $u \geq v$ in B_1 and $u(0) = v(0) = 0$. If $\phi(0^+, v) > 1$ or $v \equiv 0$, then $u \equiv v$.*

Proof. Assume by contradiction that $u \not\equiv v$. Then, $u > v$ in $\{x_{n+1} > 0\}$, and by the Hopf lemma $\partial_{n+1}(u - v)(0, 0^+) > 0$. On the other hand, since $\phi(0^+, v) > 1$ or $v \equiv 0$, $\nabla v(0) = 0$, and it follows that $\partial_{n+1} u(0, 0^+) > 0$, and since $\Delta u = 2\partial_{n+1} u \mathcal{H}^n|_{\{x_{n+1}=0\}}$ distributionally, this contradicts the fact that $\Delta u \leq 0$. □

The second one is the following cleaning result.

Proposition 2.2.7 ([96, Propositions 2.4 & 2.9])**.** *Let $u : B_1 \times [-1, 1] \to \mathbb{R}$ be a solution to (2.2)-(2.4). Let $\delta > 0$ small, and let $x_0 \in B_{1-\delta} \cap \Gamma_{\geq\kappa}(u(\cdot, t_0))$. Then, there exists $\rho > 0$ such that*

$$\{(x, t) \in B_\rho(x_0) \times [-1, 1] : t > t_0 + C|x - x_0|^{\kappa-1}\} \cap \{u = 0\} \cap \{x_{n+1} = 0\} = \emptyset,$$

for some constant C depending only on n, κ and δ. Moreover, if $\kappa = 2$, for every $\varepsilon > 0$ there exists $\rho > 0$ such that

$$\{(x, t) \in B_\rho(x_0) \times [-1, 1] : t > t_0 + C|x - x_0|^{2-\varepsilon}\} \cap \{(x, t) : x \in \Gamma_2(u(\cdot, t))\} = \emptyset,$$

for some constant C depending only on n and ε.

2.2.2 Quadratic points

Given u a solution to (2.2) and a singular point $x_0 \in \Gamma_2(u)$, we denote by p_{2,x_0} the first blow-up[1] of u at x_0,

$$p_{2,x_0} := \lim_{r\to 0} \frac{u(x_0 + r\cdot)}{r^2}. \tag{2.6}$$

This expression is uniquely defined by [110, Theorem 1.3.6 or Theorem 1.5.4], and $p_{2,x_0} \equiv 0$ if and only if $x_0 \in \Gamma_{>2}(u)$ (by [110, Lemmas 1.5.1 and 1.5.2]). The blow-up p_{2,x_0} belongs to the set of homogeneous quadratic harmonic even polynomials that are nonnegative on the thin space, i.e.

$$\mathcal{P}_2 := \{p : \Delta p = 0,\ x \cdot \nabla p = 2p,\ p(x', 0) \geq 0,\ p(x', x_{n+1}) = p(x', -x_{n+1})\}.$$

Notice how $p = 0$ also belongs to $\mathcal{P}_2$.

The following proposition will allow us to perform a second blow-up at the points of frequency 2 to attain a finer understanding of singular points:

[1] Observe that these are not rescalings that preserve the $L^2(\partial B_1)$ norm (cf. the sequence (2.3)). In fact, at singular points both types of rescalings coincide up to a multiplicative constant. By rescaling directly by r^2 we obtain the first order expansion of u, that is, $u(x_\circ + \cdot) = p_{2,x_\circ}(x) + o(|x|^2)$.

Proposition 2.2.8 ([93, Proposition 2.2]). *Let u be a solution to (2.2), and assume that $0 \in \Gamma_{\geq 2}(u)$ (i.e. $\phi(0^+, u) \geq 2$). Let $p \in \mathcal{P}_2$ and let $w := u - p$. Then, the function $r \mapsto \phi(r, w)$ is nondecreasing, and its derivative satisfies*

$$\phi'(r, w) \geq \frac{2}{r} \left(\frac{\int_{B_1} w_r \Delta w_r}{\int_{\partial B_1} w_r^2} \right)^2,$$

with $w_r(x) := w(rx)$. Moreover, $\phi(0^+, w) \geq 2$.

Proof. This result corresponds to [93, Proposition 2.2] in combination with the computations inside its proof. □

The following lemma asserts that the L^2 rate of growth of $u - p$ can be estimated by its frequency (cf. Lemma 2.2.5).

Lemma 2.2.9. *Let $u : B_1 \to \mathbb{R}$ be a solution to (2.2), and assume that $0 \in \Gamma_2(u)$ (i.e. $\phi(0^+, u) = 2$). Let $p \in \mathcal{P}_2$. Suppose that for $0 < r < R < 1$ we have $\underline{\lambda} \leq \phi(r, u - p) \leq \phi(R, u - p) \leq \overline{\lambda}$. Then, for any given $\delta > 0$,*

$$\left(\frac{R}{r}\right)^{2\underline{\lambda}} \leq \frac{H(R, u-p)}{H(r, u-p)} \leq C_\delta \left(\frac{R}{r}\right)^{2\overline{\lambda}+\delta},$$

where C_δ depends only on δ, $\overline{\lambda}$, and the dimension.

Proof. First, we define $w := u - p$, $w_r := w(r\cdot)$, and

$$F(r, w) := \frac{r^{1-n} \int_{B_r} w \Delta w}{r^{-n} \int_{\partial B_r} w^2} = \frac{\int_{B_1} w_r \Delta w_r}{\int_{\partial B_1} w_r^2}.$$

Since $p \geq 0$ on the thin space, and $\Delta u = 0$ outside of it, $w \Delta w = -p \Delta u \geq 0$.

Observe that

$$H'(r, w) = \frac{2}{r} \int_{B_1} |\nabla w_r|^2 + \frac{2}{r} \int_{B_1} w_r \Delta w_r \quad \Rightarrow \quad \frac{H'(r, w)}{H(r, w)} = \frac{2}{r} \left(\phi(r, w) + F(r)\right).$$

Integrating, we get

$$\ln\left(\frac{H(R, w)}{H(r, w)}\right) = \int_r^R \frac{2}{\rho} \left(\phi(\rho, w) + F(\rho, w)\right) \mathrm{d}\rho.$$

On the one hand, since $F(\rho, w) \geq 0$ and ϕ is nondecreasing (by Proposition 2.2.8),

$$\ln\left(\frac{H(R, w)}{H(r, w)}\right) \geq \int_r^R \frac{2}{\rho} \phi(\rho, w) \mathrm{d}\rho \geq 2\underline{\lambda} \ln(R/r),$$

and the inequality in the left follows. On the other hand, using Proposition 2.2.8,

$$\int_r^R F(\rho, w) \frac{\mathrm{d}\rho}{\rho} \leq \left(\int_r^R F(\rho, w)^2 \frac{\mathrm{d}\rho}{\rho}\right)^{1/2} \left(\int_r^R \frac{\mathrm{d}\rho}{\rho}\right)^{1/2} \leq \left(\frac{\overline{\lambda} - \underline{\lambda}}{2}\right)^{1/2} \ln(R/r)^{1/2},$$

and then

$$\ln\left(\frac{H(R, w)}{H(r, w)}\right) = \int_r^R \frac{2}{\rho} \left(\phi(\rho, w) + F(\rho, w)\right) \mathrm{d}\rho \leq 2\overline{\lambda} \ln(R/r) + C \ln(R/r)^{1/2},$$

so that the conclusion follows by the estimate $\sqrt{t} \leq \delta t + C_\delta$. □

By means of Proposition 2.2.8, quadratic free boundary points can be further stratified in terms of a second blow-up. That is, if $x_0 \in \Gamma_2(u)$, we define the second blow-up sequence

$$\tilde{w}_r := \frac{u(x_0 + r\cdot) - p_{2,x_0}(r\cdot)}{\|u(x_0 + r\cdot) - p_{2,x_0}(r\cdot)\|_{L^2(\partial B_1)}},$$

which converges to a λ-homogeneous function with $\lambda = \phi(0^+, u(x_0 + \cdot) - p_{2,x_0})$, up to a subsequence, thanks to the monotonicity of ϕ along $u - p$ given by Proposition 2.2.8:

Proposition 2.2.10 ([93, Proposition 3.2])**.** *For every sequence $r_j \downarrow 0$, there is a subsequence $r_{j_l} \downarrow 0$ such that $\tilde{w}_{r_{j_l}} \rightharpoonup q$ in H^1loc, and $q \not\equiv 0$ is a λ-homogeneous harmonic polynomial, with $\lambda = \phi(0^+, u(x_0 + \cdot) - p_{2,x_0}) \in \{2, 3, 4, \ldots\}$.*

Then, we define the ordinary and anomalous quadratic points as follows:

$$\begin{aligned} \Gamma_2^{\mathrm{o}}(u) &:= \{x_0 \in \Gamma_2(u) : \phi(0^+, u(x_0 + \cdot) - p_{2,x_0}) \geq 3\} \\ \Gamma_2^{\mathrm{a}}(u) &:= \{x_0 \in \Gamma_2(u) : \phi(0^+, u(x_0 + \cdot) - p_{2,x_0}) = 2\}, \end{aligned} \tag{2.7}$$

and we define the sets $\mathbf{\Gamma}_2^{\mathrm{o}}$ and $\mathbf{\Gamma}_2^{\mathrm{a}}$ analogously for a family of solutions (cf. (2.5)). Ordinary quadratic points are called *generic quadratic points* in [93], but we have decided to change the terminology in order to avoid confusion.

The second blow-up satisfies the following orthogonality property with the first one, coming from an optimality condition:

Lemma 2.2.11 ([93, Lemma 3.3])**.** *Let u be a solution to (2.2) with $0 \in \Gamma_2(u)$. Let $p_2 \in \mathcal{P}_2$ be the blow-up of u at 0, and let q be a second blow-up as introduced in Proposition 2.2.10. Then,*

$$\int_{\partial B_1} p_2 q = 0$$

and

$$\int_{\partial B_1} pq \leq 0 \quad \forall p \in \mathcal{P}_2.$$

2.2.3 Cubic points

We will take advantage of the following recently improved convergence to the cubic blow-up:

Theorem 2.2.12 ([175, Theorem 1.1])**.** *Let u be a solution to (2.2) with $0 \in \Gamma_3(u)$ and $\|u\|_{L^\infty(B_1)} \leq 1$. Then, there exists a 3-homogeneous solution to (2.2), p_3, such that*

$$\|u - p_3\|_{L^\infty(B_r)} \leq Cr^{3+\alpha},$$

for some $C, \alpha > 0$ depending only on n.

We will also use the following characterization of global cubic solutions.

Lemma 2.2.13 ([102, Lemma 5.2])**.** *Let p_3 be a 3-homogeneous solution to (2.2). Then,*

$$p_3(x) = |x_{n+1}|(ax_{n+1}^2 - x' \cdot Ax'),$$

where $a \geq 0$, A is symmetric and nonnegative definite, and $a = \operatorname{Tr} A$.

2.2.4 Geometric measure theory tools

We will use the following Reifenberg-type result using the frequency function ϕ as f, to perform dimension reduction arguments only at the points of continuity of ϕ.

Lemma 2.2.14 ([102, Lemma 7.3]). *Let $E \subset \mathbb{R}^n$, and $f : E \to \mathbb{R}$. Assume that, for any $\varepsilon > 0$ and $x \in E$, there exists $\rho > 0$ such that, for all $r \in (0, \rho)$,*

$$E \cap \overline{B_r(x)} \cap f^{-1}\Big([f(x) - \rho, f(x) + \rho]\Big) \subset \{y : \mathrm{dist}(y, \Pi_{x,r}) \le \varepsilon r\},$$

for some m-dimensional plane $\Pi_{x,r}$ passing through x (possibly depending on r). Then, $\dim_{\mathcal{H}}(E) \le m$.

We will also use the following abstract proposition in the proof of our main result, Theorem 2.1.2, in order to bound the sizes of each of the subsets of the free boundary.

Proposition 2.2.15 ([102, Corollary 7.8]). *Consider the family $\{E_t\}_{t\in[-1,1]}$ with $E_t \subset \mathbb{R}^n$, and let us denote $E := \bigcup_{t\in[-1,1]} E_t$.*

Let $1 \le \beta \le n$, and assume that the following holds:

- $\dim_{\mathcal{H}} E \le \beta$,
- *for all $\varepsilon > 0$, $t_0 \in [-1, 1]$, and $x_0 \in E_{t_0}$, there exists $\rho > 0$ such that*

$$B_r(x_0) \cap E_t = \emptyset,$$

  for all $r \in (0, \rho)$ and $t > t_0 + r^{\gamma-\varepsilon}$.

Then,

(a) If $\gamma > \beta$, $\dim_{\mathcal{H}}(\{t : E_t \neq \emptyset\}) \le \beta/\gamma$.

(b) If $\gamma \le \beta$, $\dim_{\mathcal{H}}(E_t) \le \beta - \gamma$, for $\mathcal{H}^1$-a.e. $t \in [-1, 1]$.

2.2.5 Sketch of the proof

The proof is done by combining the ideas and techniques from [102] with the results in [65, 93, 96, 175].

The two key parts of our strategy are dimension reduction arguments for families of solutions and cleaning lemmas combined with Proposition 2.2.15. We then apply the two steps to different subsets of the free boundary, using the following stratification:

$$\mathrm{Deg}(u) = \Gamma_2^{\mathrm{o}}(u) \cup \Gamma_2^{\mathrm{a}}(u) \cup \Gamma_3(u) \cup \Gamma_{\ge 7/2}(u) \cup \Gamma_*(u).$$

First, given a family of solutions $u : B_1 \times [-1, 1] \to \mathbb{R}$ to (2.2)-(2.4), using dimension reduction arguments one can compute the maximum total dimension of each of the five sets for all the solutions of the family at the same time, see Propositions 2.3.1 and 2.4.3. Here, monotonicity is key to get the same results as one would get for a single solution.

Then, for each type of points we use that if $x_0 \in \Gamma(u(\cdot, t_0))$, there exists some $r_0 > 0$ such that u is positive (or identically zero, depending on the case) in one of the following sets

$$\{x \in B_{r_0} : |x - x_0|^\gamma < t - t_0\} \quad \text{or} \quad \{x \in B_{r_0} : |x - x_0|^\gamma < t_0 - t\},$$

and hence there are no other free boundary points there. This is done via an expansion of the solution at x_0 and comparison arguments. The novel results in this step are Propositions 2.4.1 and 2.5.1, that deal with quadratic and cubic points, respectively.

Finally, applying Proposition 2.2.15 one can get an estimate on the size of each of the degenerate strata for almost every solution. For $n \geq 4$, the situation can be summarized as follows, where $\alpha, \gamma \in (0, 1)$ are dimensional constants, and $\varepsilon > 0$ is an arbitrarily small number.

Set	$\dim_{\mathcal{H}} \mathbf{\Gamma}$	Cleaning exponent	Generic[2] $\dim_{\mathcal{H}} \Gamma$
$\Gamma_2^{\rm o}$	$n-1$	$3-\varepsilon$	$n-4$
$\Gamma_2^{\rm a}$	$n-2$	$2-\varepsilon$	$n-4$
Γ_3	$n-1$	$2+\gamma$	$n-3-\gamma$
$\Gamma_{\geq 7/2}$	$n-1$	$5/2-\varepsilon$	$n-7/2$
Γ_*	$n-2$	$1+\alpha$	$n-3-\alpha$

For $n = 2$ and $n = 3$, the conclusion is that, generically, the free boundary contains no degenerate points.

2.3 Dimensional bounds for $\Gamma_{\geq 2}$ and Γ_*

First, we will estimate the size of the sets $\mathbf{\Gamma}_{\geq 2}$ and $\mathbf{\Gamma}_*$ with a dimension reduction argument (recall (2.5)), taking advantage of the fact that the possible global homogeneous solutions of the Signorini problem are completely classified in low dimensions.

In particular, the goal of this section is to prove the following result:

Proposition 2.3.1. *Let $u : B_1 \times [-1, 1] \to \mathbb{R}$ be a solution to (2.2)-(2.4). Then,*

(a) $\dim_{\mathcal{H}}(\mathbf{\Gamma}_{\geq 2}) \leq n - 1$ *if $n \geq 2$, and $\mathbf{\Gamma}_{\geq 2}$ is discrete if $n = 1$.*

(b) $\dim_{\mathcal{H}}(\mathbf{\Gamma}_*) \leq n - 2$ *if $n \geq 3$, $\mathbf{\Gamma}_*$ is discrete if $n = 2$, and it is empty if $n = 1$.*

In order to do it, we first show the following lemma (cf. [102, Lemma 6.5]).

Lemma 2.3.2. *Let $u : B_1 \times [-1, 1] \to \mathbb{R}$ be a solution to (2.2)-(2.4), with $0 \in \Gamma_{\geq 2}(u(\cdot, 0))$. Let $x_k \in \mathbf{\Gamma}_{\geq 2}$ satisfy $|x_k| \leq r_k$, with $r_k \downarrow 0$, $t_k := \tau(x_k) \to 0$, and assume that*

$$\tilde{u}_{r_k} := \frac{u(r_k \cdot, 0)}{\|u(r_k \cdot, 0)\|_{L^2(\partial B_1)}} \rightharpoonup q \text{ in } H^1_{\rm loc}(\mathbb{R}^{n+1}), \quad y_k := \frac{x_k}{r_k} \to y_\infty \neq 0, \text{ and } \kappa_k \to \kappa,$$

where

$$\kappa_k := \phi(0^+, u(x_k + \cdot, t_k)), \quad \kappa := \phi(0^+, u(\cdot, 0)),$$

and $q \not\equiv 0$ is a κ-homogeneous solution to (2.2).

Then, q is translation invariant in the direction y_∞.

[2] In the sense of Remark 2.1.4.

Proof. Let us define $w_k := u(x_k + r_k\cdot, t_k)$ and $w_{k,0} := u(x_k + r_k\cdot, 0)$ so that, for each $k \in \mathbb{N}$, they are ordered in $B_{1/(2r_k)}$ (that is, either $w_k \geq w_{k,0}$ or $w_k \leq w_{k,0}$ in $B_{1/(2r_k)}$). Observe that, by assumption, since $\tilde{u}_{r_k} \rightharpoonup q$ and $q \not\equiv 0$,

$$\frac{w_{k,0}}{\|w_{k,0}\|_{L^2(\partial B_1)}} = \frac{\tilde{u}_{r_k}(y_k + \cdot)}{\|\tilde{u}_{r_k}(y_k + \cdot)\|_{L^2(\partial B_1)}} \rightharpoonup \frac{q(y_\infty + \cdot)}{\|q(y_\infty + \cdot)\|_{L^2(\partial B_1)}}$$

weakly in H^1loc. We now divide the proof into two steps.

Step 1. We first prove that, up to a subsequence,

$$\tilde{w}_k := \frac{w_k}{\|w_k\|_{L^2(\partial B_1)}} \to Q \quad \text{locally uniformly,}$$

for some Q a global κ-homogeneous solution to the Signorini problem.

Indeed, by the upper semicontinuity and monotonicity of ϕ, and the fact that $\kappa_k \to \kappa$, for all $\delta > 0$ there exist $r_\delta > 0$ and $k_\delta \in \mathbb{N}$ such that

$$\phi(r, u(x_k + \cdot, t_k)) \in (\kappa - \delta, \kappa + \delta) \quad \forall r \in (0, r_\delta), \quad \forall k \geq k_\delta,$$

and hence

$$\phi(r, \tilde{w}_k) \in (\kappa - \delta, \kappa + \delta) \quad \forall r \in (0, r_\delta/r_k), \quad \forall k \geq k_\delta.$$

In particular, by Lemma 2.2.5,

$$H(R, \tilde{w}_k) \leq R^{2\kappa+2\delta} H(1, \tilde{w}_k) = R^{2\kappa+2\delta} \quad \forall R \in [1, r_\delta/r_k), \quad \forall k \geq k_\delta,$$

maybe with a smaller $r_\delta > 0$ and larger k_δ. Combined with interior Lipschitz estimates [7, Theorem 1], this implies that $\tilde{w}_k \to Q$ locally uniformly, up to a subsequence, for some Q a global solution to the thin obstacle problem. Moreover, thanks to the uniform $C^{1,1/2}$ estimates for solutions [7] we also have that $\phi(r, \tilde{w}_k) \to \phi(r, Q)$ as $k \to \infty$ for each $r > 0$ fixed (observe that $|\partial_{n+1}\tilde{w}_k|^2$ is $C^{1/2}$), and therefore

$$\phi(r, Q) \in [\kappa - \delta, \kappa + \delta] \quad \forall r > 0.$$

Since this holds for any $\delta > 0$, Lemma 2.2.4 yields that Q is κ-homogeneous.

Step 2. We now show that $y_\infty \cdot \nabla q$ has a constant sign and deduce that $y_\infty \cdot \nabla q = 0$.

Let $\hat{\varepsilon}_k := \|w_k\|_{L^2(\partial B_1)} + \|w_{k,0}\|_{L^2(\partial B_1)}$. By the first observation we have

$$\hat{w}_{k,0} := w_{k,0}/\hat{\varepsilon}_k \rightharpoonup bq(y_\infty + \cdot) =: \hat{Q}_0 \text{ weakly in } H^1\text{loc}$$

for some $b \in [0, 1]$. Moreover, by Step 1 and up to a subsequence,

$$\hat{w}_k := w_k/\hat{\varepsilon}_k \to aQ =: \hat{Q} \quad \text{locally uniformly,}$$

with $a \in [0, 1]$.

We cannot have $a = b = 0$, because it contradicts the fact that $\|\hat{Q}\|_{L^2(\partial B_1)} + \|\hat{Q}_0\|_{L^2(\partial B_1)} = 1$. Suppose now that $a = 0$. Then, for each $k \in \mathbb{N}$, $\hat{w}_k$ and $\hat{w}_{k,0}$ are ordered in $B_{1/(2r_k)}$, and therefore $\hat{Q}_0$ and $\hat{Q}$ are ordered in $\mathbb{R}^{n+1}$ (that is, either $\hat{Q}_0 \geq \hat{Q} \equiv 0$ or $\hat{Q}_0 \leq \hat{Q} \equiv 0$ in $\mathbb{R}^{n+1}$). Since q (and then $\hat{Q}_0$) is a global solution with homogeneity $\kappa \geq 2$, by Lemma 2.2.1 it cannot have constant sign, a contradiction. The same argument with Q gives that b cannot be zero. Hence, a and b are both positive.

If we assume without loss of generality that $\hat{Q} \geq \hat{Q}_0$ and let $z = \lambda x$, by homogeneity we have

$$aQ(x) \geq bq(y_\infty + x) \quad \Rightarrow \quad aQ(z) \geq bq(\lambda y_\infty + z) \;\; \forall \lambda > 0 \quad \Rightarrow \quad aQ \geq bq.$$

Since aQ and bq are ordered global solutions of (2.2) with homogeneity greater than 1, they are equal by Lemma 2.2.6. It follows that

$$bq = aQ \geq bq(y_\infty + \cdot),$$

and by homogeneity again (since $b > 0$)

$$q \geq q(\lambda y_\infty + \cdot) \quad \forall \lambda > 0.$$

Thus, $y_\infty \cdot \nabla q \leq 0$, and applying Lemma 2.2.1(c), q is invariant in the y_∞ direction. □

We can now give the proof of Proposition 2.3.1.

Proof of Proposition 2.3.1. (a) We will apply Proposition 2.2.14 to the set $\mathbf{\Gamma}_{\geq 2}$ with the function $f : \mathbf{\Gamma}_{\geq 2} \to \mathbb{R}$ given by

$$f(x_0) = \phi(0^+, u(\cdot, \tau(x_0))).$$

To obtain the desired result, thanks to Lemma 2.2.14 it suffices to prove the following: for all $x_0 \in \mathbf{\Gamma}_{\geq 2}$ and for all $\varepsilon > 0$, there exists $\rho > 0$ such that for all $r \in (0, \rho)$,

$$B_r(x_0) \cap \mathbf{\Gamma}_{\geq 2} \cap f^{-1}\Big([f(x_0) - \rho, f(x_0) + \rho]\Big) \subset \{y : \operatorname{dist}(y, \Pi_{x,r}) \leq \varepsilon r\},$$

where $\Pi_{x,r}$ is a $(n-1)$-dimensional plane passing through x_0.

Assume without loss of generality that $x_0 = 0$ and $\tau(x_0) = 0$, and let us prove the statement by contradiction. If such $\rho > 0$ did not exist for some $\varepsilon_0 > 0$, then we would have sequences $r_k \downarrow 0$ and $x_k^{(j)} \in \mathbf{\Gamma}_{\geq 2} \cap B_{r_k}$, $1 \leq j \leq n$, such that

$$y_k^{(j)} := x_k^{(j)}/r_k \to y_\infty^{(j)} \in \overline{B_1}, \quad \dim(\operatorname{span}(y_\infty^{(1)}, \ldots, y_\infty^{(n)})) = n, \quad |f(x_k^{(j)}) - f(0)| \downarrow 0.$$

Let $\tilde{u}_r := u(r\cdot)/H(r,u)^{1/2}$. Then, by [11, Section 4] $\tilde{u}_r \rightharpoonup q$ along a subsequence, where q is a nonzero κ-homogeneous global solution to the Signorini problem (2.2). Also, since $x_0 \in \mathbf{\Gamma}_{\geq 2}$, $\kappa \geq 2$.

Applying Lemma 2.3.2 to the sequences $(x_k^{(j)}, \tau(x_k^{(j)}))$ we deduce that q is translation invariant in the n linearly independent directions $y_\infty^{(j)}$, $1 \leq j \leq n$. It follows that q is a one dimensional nonzero κ-homogeneous solution to Signorini, with $\kappa \geq 2$, which contradicts the fact that the only possible homogeneities in dimension one are 0 and 1.

(b) Repeating the arguments in (a), but with $1 \leq j \leq n-1$ instead, we end up with a nonzero κ-homogeneous two dimensional solution to Signorini, but since $x_0 \in \mathbf{\Gamma}_*$, $\kappa \notin \{1, \frac{3}{2}, 2, 3, \frac{7}{2}, 4, 5, \ldots\}$, contradicting that these are the only possible homogeneities in dimension 2. □

2.4 Quadratic points

2.4.1 Ordinary quadratic points

If the next term of the expansion at a quadratic point is at least cubic (that is, we are at an ordinary quadratic point, (2.7)), we can adapt the arguments in [102, Section 9] to improve the cleaning rate up to $3-\varepsilon$. Hence, we show:

Proposition 2.4.1. *Let $u : B_1 \times [-1,1] \to \mathbb{R}$ be a solution to (2.2)-(2.4). Assume that $0 \in \Gamma_2^{\circ}(u(\cdot,0))$.*

Then, for all $\varepsilon > 0$ there exists $\rho > 0$ such that

$$\{(x,t) \in B_\rho \times [0,1] : t > |x|^{3-\varepsilon}\} \cap \{u = 0\} \cap \{x_{n+1} = 0\} = \emptyset.$$

In order to prove Proposition 2.4.1, we first show the following auxiliary lemma.

Lemma 2.4.2. *Let $u : B_1 \times [-1,1] \to \mathbb{R}$ be a solution to (2.2)-(2.4), with $0 \in \Gamma_2(u(\cdot,0))$. Let $D_r := \partial B_r \cap \{|x_{n+1}| > r/2\}$. Then, for every $\varepsilon > 0$,*

$$\min_{D_r} h_t := \min_{D_r} [u(\cdot,t) - u(\cdot,0)] \geq c_\varepsilon r^\varepsilon t, \quad \forall r \in (0,\rho_\varepsilon), \quad \forall t \in [0,1],$$

for some $c_\varepsilon, \rho_\varepsilon > 0$.

Proof. By [110, Theorem 1.3.6]),

$$u(x,0) = p(x) + o(|x|^2),$$

for some nonzero $p \in \mathcal{P}_2$. Therefore, for all $\delta > 0$ there exists $r_\delta > 0$ such that for all $\rho \in (0, 2r_\delta)$,

$$B_1 \cap \{u(\rho\cdot,0) = 0\} \cap \{x_{n+1} = 0\} \subset C_\delta := \left\{ x \in \mathbb{R}^{n+1} : \operatorname{dist}\left(\frac{x}{|x|}, \{p=0\} \cap \{x_{n+1} = 0\}\right) < \delta \right\}.$$

Indeed, let m be the minimum of p in $(\partial B_1 \cap \{x_{n+1} = 0\}) \setminus C_\delta$. Since $p \geq 0$ on the thin space, $m > 0$. Now, choosing r_δ small enough, for all $\rho < r_\delta$,

$$u(\rho x, 0) \geq p(\rho x) - \frac{m}{2}\rho^2|x|^2 = \rho^2|x|^2 \left(p\left(\frac{x}{|x|}\right) - \frac{m}{2}\right) > 0,$$

for all $x \in (B_1 \cap \{x_{n+1} = 0\}) \setminus C_\delta$.

Let now $\varphi_\delta := |x|^{\mu(\delta)}\Phi_\delta(x/|x|)$, where $\Phi_\delta \geq 0$ is the first eigenfunction of the spherical Laplacian on $\partial B_1 \setminus C_\delta$, and $\mu(\delta)$ is chosen so that φ_δ is harmonic when positive. Then, φ_δ is a positive harmonic function defined in $\mathbb{R}^n \setminus C_\delta$ vanishing on ∂C_δ.

Since $p \not\equiv 0$ and p is a homogeneous quadratic polynomial nonnegative on the thin space, $\{p = 0\} \cap \{x_{n+1} = 0\}$ is a linear space of dimension at most $n-1$, and in particular has zero harmonic capacity. Therefore, as $\delta \to 0$, $\mu(\delta) \to 0$, and we can choose δ such that $\mu(2\delta) < \varepsilon$. Moreover, choosing $\delta < \frac{1}{4}$, D_{r_δ} and $C_{2\delta}$ are disjoint.

Notice that $h_t = u(\cdot,t) - u(\cdot,0)$ is harmonic in $\{u(\cdot,0) > 0\}$ and in $B_1 \setminus \{x_{n+1} = 0\}$. In particular, h_t is harmonic in

$$(B_1 \setminus \{x_{n+1} = 0\}) \cup (B_{2r_\delta} \cap \{x_{n+1} = 0\} \setminus C_\delta).$$

Hence, using the monotonicity assumption (2.4) and the interior Harnack, there exists $c_\delta > 0$ such that

$$h_t \geq c_\delta t \text{ on } \partial B_{r_\delta} \setminus C_{2\delta}.$$

Then, we can use

$$w_t := c_\delta t \frac{\varphi_{2\delta}}{\|\varphi_{2\delta}\|_{L^\infty(\partial B_{r_\delta})}}$$

as a lower barrier in $B_{r_\delta} \setminus C_{2\delta}$ because $h_t \geq w_t$ in $\partial B_{r_\delta} \setminus C_{2\delta}$ by construction, and $h_t \geq 0$ and $w_t = 0$ on $\partial C_{2\delta}$.

Hence,

$$\min_{D_r} h_t \geq \min_{D_r} w_t = cr^{\mu(2\delta)} t \geq cr^\varepsilon t \quad \forall r \in (0, r_\delta),$$

as we wanted to see. □

By means of the previous result, we can now prove the improved cleaning for the ordinary quadratic points.

Proof of Proposition 2.4.1. By the definition of $\mathbf{\Gamma}_2^{\rm o}$, there exists a harmonic quadratic polynomial $p \in \mathcal{P}_2$ such that

$$|r^{-2}u(r\cdot, 0) - p| \leq Cr \text{ in } B_1, \quad \forall r \in (0,1),$$

Let us then bound $v(x) := r^{-2}u(rx, t)$. By Lemma 2.4.2 and the previous estimates, taking $t \geq r^{3-2\varepsilon}$,

$$v(x) \geq p(x) - Cr + c_\varepsilon r^{\varepsilon-2} t \chi_{\{|x_{n+1}|>1/2\}} \geq p(x) - Cr + c_\varepsilon r^{1-\varepsilon} \chi_{\{|x_{n+1}|>1/2\}} \text{ on } \partial B_1.$$

Let φ be a harmonic function in B_1 with boundary data $\varphi = \chi_{\{|x_{n+1}|>1/2\}}$ on ∂B_1. Then, since v is superharmonic and p is harmonic,

$$v(x) \geq p(x) - Cr + c_\varepsilon r^{1-\varepsilon} \varphi \text{ in } B_1,$$

and using that $\varphi \geq c(n) > 0$ in $B_{1/2}$,

$$v \geq p - Cr + c_\varepsilon c(n) r^{1-\varepsilon} > 0 \text{ on } B_{1/2} \cap \{x_{n+1} = 0\},$$

for sufficiently small r, using that $p \geq 0$ on the thin space. □

2.4.2 Anomalous quadratic points

Now we consider the points in the set $\mathbf{\Gamma}_2^{\rm a}$ (see (2.7)). We will use a dimension reduction argument to show that $\dim_{\mathcal{H}}(\mathbf{\Gamma}_2^{\rm a}) \leq n-2$. Hence, in this subsection we will prove the following proposition.

Proposition 2.4.3. *Let $u : B_1 \times [-1, 1] \to \mathbb{R}$ be a solution to (2.2)-(2.4). Then,* $\dim_{\mathcal{H}}(\mathbf{\Gamma}_2^{\rm a}) \leq n - 2$ *if $n \geq 3$, $\mathbf{\Gamma}_2^{\rm a}$ is discrete if $n = 2$, and it is empty if $n = 1$.*

The following lemmas are analogous to the first part of [102, Section 6] combined with results from [65, 93, 96]. The first one is about the continuity of the first and second blow-ups on the set $\mathbf{\Gamma}_2$.

Lemma 2.4.4. *Let $u : B_1 \times [-1,1] \to \mathbb{R}$ be a solution to (2.2)-(2.4), and let us denote by $p_{2,x}$ the blow-up of $u(\cdot,\tau(x))$ at $x \in \mathbf{\Gamma}_{\geq 2}$ according to (2.6); in particular, $p_{2,x} \equiv 0$ if and only if $x \in \mathbf{\Gamma}_{>2}$. Then:*

(a) For all $\rho < 1$, $\mathbf{\Gamma}_{\geq 2} \cap \overline{B_\rho}$ is closed. Moreover, given a convergent sequence $\{x_k\} \subset \mathbf{\Gamma}_{\geq 2} \cap \overline{B_\rho}$, $x_k \to x_\infty$,

$$p_{2,x_k} \to p_{2,x_\infty},$$

where $p_{2,x_\infty} \equiv 0$ if $x_\infty \in \mathbf{\Gamma}_{>2}$.

(b) The frequency function

$$\mathbf{\Gamma}_{\geq 2} \ni x_0 \mapsto \phi\Big(0^+, u(x_0 + \cdot, \tau(x_0)) - p_{2,x_0}\Big)$$

is upper semicontinuous.

Proof. (a) We first show that if $x_k \in \mathbf{\Gamma}_{\geq 2}$ and $x_k \to x_\infty$, then $x_\infty \in \mathbf{\Gamma}_{\geq 2}$. Notice that $t_k := \tau(x_k) \to t_\infty := \tau(x_\infty)$ by Proposition 2.2.3. Now, by [65, Proposition 7.1] (or by the frequency gap [65, Theorem 4] if $x_k \in \mathbf{\Gamma}_{>2}$) we have

$$\|u(x_k + \cdot, t_k) - p_{2,x_k}\|_{L^\infty(B_r)} \leq r^2\omega(r), \quad \forall r > 0,$$

where ω is a universal modulus of continuity.

Then, $p_{2,x_k} \to P$ up to a subsequence for some harmonic 2-homogeneous polynomial P and, by Proposition 2.2.3, $u(x_k + \cdot, t_k) \to u(x_\infty + \cdot, t_\infty)$ in C^0. Therefore,

$$\|u(x_\infty + \cdot, t_\infty) - P\|_{L^\infty(B_r)} \leq r^2\omega(r), \quad \forall r > 0.$$

It follows that $x_\infty \in \mathbf{\Gamma}_{\geq 2}$ and that $p_{2,x_\infty} = P$. Finally, the estimate can only hold for one unique P, and a posteriori we deduce that for any other subsequence, $p_{2,x_{k_j}} \to P$ up to a subsequence again.

(b) First, we consider the function $\mathbf{\Gamma}_{\geq 2} \ni x_0 \mapsto \phi(r, u(x_0 + \cdot, \tau(x_0)) - p_{2,x_0})$ for a fixed $r > 0$,

$$\phi(r, u(x_0 + \cdot, \tau(x_0)) - p_{2,x_0}) = r\frac{\int_{B_r} |\nabla u(x_0 + \cdot, \tau(x_0)) - \nabla p_{2,x_0}|^2}{\int_{\partial B_r} (u(x_0 + \cdot, \tau(x_0)) - p_{2,x_0})^2}.$$

Given a convergent sequence $x_k \in \mathbf{\Gamma}_{\geq 2}$, $x_k \to x_\infty$, using (a) the terms involving the second order polynomial converge. Then, $u(x_k + \cdot, \tau(x_k)) \to u(x_\infty + \cdot, \tau(x_\infty))$ in L^∞ by the second part of Proposition 2.2.3. Thus, the quotient is continuous because of the uniform $C^{1,1/2}$ estimates for $u(\cdot, t)$ [7] (observe that $|\partial_{n+1}u(x_0 + \cdot, \tau(x_0)) - \partial_{n+1}p_{2,x_0}|^2 = |\partial_{n+1}u(x_0 + \cdot, \tau(x_0))|^2$ is $C^{1/2}$ in B_r).

Our desired result now follows by taking the infimum over $r > 0$ of the family of continuous functions $\mathbf{\Gamma}_{\geq 2} \ni x_0 \mapsto \phi(r, u(x_0 + \cdot, \tau(x_0)) - p_{2,x_0})$ (this is an increaing family in $r > 0$, by Proposition 2.2.8). □

Then, we show that points in $\mathbf{\Gamma}_2$ only accumulate in the directions of the null space of the blow-up.

Lemma 2.4.5. *Let $u : B_1 \times [-1,1] \to \mathbb{R}$ be a solution to (2.2)-(2.4), and let $0 \in \Gamma_2(u(\cdot,0))$. Let $x_k \in \mathbf{\Gamma}_2$ satisfy $|x_k| \downarrow 0$ and $t_k := \tau(x_k) \downarrow 0$. Let $p_{2,k} := p_{2,x_k}$. Then, $p_{2,k} \to p_2$, with p_2 the blow-up of $u(\cdot,0)$ at 0, and we have*

$$\left\| p_{2,k} - p_2\left(\frac{x_k}{|x_k|} + \cdot\right)\right\|_{L^\infty(B_1)} \le C\omega(2|x_k|),$$
$$\|p_{2,k} - p_2\|_{L^\infty(B_1)} \le C\omega(2|x_k|),$$

where ω is a universal modulus of continuity, and

$$\operatorname{dist}\left(\frac{x_k}{|x_k|}, \{p_2 = 0\} \cap \{x_{n+1} = 0\}\right) \to 0 \quad \textit{as} \quad k \to \infty.$$

Proof. By Lemma 2.4.4 (a), $p_{2,k} \to p_2$, up to a subsequence. Let $r_k = |x_k|$, so that by [65, Proposition 7.1] we have

$$\|r_k^{-2}u(x_k + r_k x, t_k) - p_{2,k}(x)\|_{L^\infty(B_2)} \le 4\omega(2r_k)$$

and

$$\|r_k^{-2}u(r_k x, 0) - p_2(x)\|_{L^\infty(B_2)} \le 4\omega(2r_k).$$

Thus, defining $y_k := x_k/|x_k|$, for all $x \in B_2$ we have the following: if $t_k \le 0$,

$$-4\omega(2r_k) + p_{2,k}(x) \le r_k^{-2}u(x_k + r_k x, t_k) \le r_k^{-2}u(x_k + r_k x, 0) \le 4\omega(2r_k) + p_2(y_k + x),$$

and if $t_k \ge 0$,

$$4\omega(2r_k) + p_{2,k}(x) \ge r_k^{-2}u(x_k + r_k x, t_k) \ge r_k^{-2}u(x_k + r_k x, 0) \ge -4\omega(2r_k) + p_2(y_k + x).$$

Assume without loss of generality that $t_k \ge 0$ and consider the function $q(x) = p_{2,k}(x) - p_2(y_k + x) + 8\omega(2r_k)$. On the one hand, q is nonnegative and harmonic in B_2. On the other hand, since $p_2(y_k + \cdot) \ge 0$ on $\{x_{n+1} = 0\}$, $q(0) \le 8\omega(2r_k)$. Then, by the Harnack inequality, $0 \le q \le C\omega(2r_k)$ in B_1.

Consequently,

$$\|p_{2,k} - p_2(y_k + \cdot)\|_{L^2(\partial B_1)} \le C\|p_{2,k} - p_2(y_k + \cdot)\|_{L^\infty(B_1)} \le C\omega(2r_k).$$

Finally, $p_{2,k} - p_2$ is 2-homogeneous and harmonic, and $p_2 - p_2(y_k + \cdot)$ is affine. Therefore, they are orthogonal. Hence, when $k \to \infty$,

$$\|p_{2,k} - p_2\|^2_{L^2(\partial B_1)} + \|p_2 - p_2(y_k + \cdot)\|^2_{L^2(\partial B_1)} = \|p_{2,k} - p_2(y_k + \cdot)\|^2_{L^2(\partial B_1)} \to 0.$$

In particular, $\|p_2 - p_2(y_k + \cdot)\|_{L^2(\partial B_1)} \to 0$, and it follows that $\operatorname{dist}(y_k, \{p_2 = 0\} \cap \{x_{n+1} = 0\}) \to 0$. □

The following auxiliary lemma plays a similar role to Lemma 2.3.2, but for the second blow-up at anomalous quadratic points.

Lemma 2.4.6. *Let $u : B_1 \times [-1,1] \to \mathbb{R}$ be a solution to (2.2)-(2.4), let $0 \in \Gamma_2^{\mathrm{a}}(u(\cdot,0))$. Let $x_k \in \mathbf{\Gamma}_2^{\mathrm{a}}$ satisfy $|x_k| \le r_k$ with $r_k \downarrow 0$ and $t_k := \tau(x_k) \to 0$. Assume that*

$$\tilde{w}_{r_k} := \frac{w(r_k \cdot)}{\|w(r_k\cdot)\|_{L^2(\partial B_1)}} \rightharpoonup q \text{ in } H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}) \text{ for } w := u(\cdot,0) - p_2, \quad y_k := \frac{x_k}{r_k} \to y_\infty,$$

where p_2 is the blow-up of $u(\cdot,0)$ at 0 and $y_\infty \neq 0$.

Then, $q(y_\infty) = 0$.

Proof. Let us define $v_k := u(x_k + r_k\cdot, t_k) - p_2(r_k\cdot) = v_k^{(1)} + v_k^{(2)} + v_k^{(3)}$, where

$$\begin{aligned} v_k^{(1)} &:= u(x_k + r_k\cdot, t_k) - u(x_k + r_k\cdot, 0),\\ v_k^{(2)} &:= u(x_k + r_k\cdot, 0) - p_2(x_k + r_k\cdot),\\ v_k^{(3)} &:= p_2(x_k + r_k\cdot) - p_2(r_k\cdot). \end{aligned}$$

Observe that $\tilde{w}_{r_k} \rightharpoonup q$, and $\|q(y_k + \cdot)\|_{L^2(\partial B_1)} \neq 0$ because q is homogeneous and nonzero by Proposition 2.2.10. Therefore,

$$\frac{v_k^{(2)}}{\|v_k^{(2)}\|_{L^2(\partial B_1)}} = \frac{w_{r_k}(y_k + \cdot)}{\|w_{r_k}(y_k+\cdot)\|_{L^2(\partial B_1)}} = \frac{\tilde{w}_{r_k}(y_k+\cdot)}{\|\tilde{w}_{r_k}(y_k+\cdot)\|_{L^2(\partial B_1)}} \rightharpoonup \frac{q(y_\infty+\cdot)}{\|q(y_\infty+\cdot)\|_{L^2(\partial B_1)}},$$

weakly in H^1loc.

On the other hand, notice that the zero level set of a nonnegative homogeneous quadratic polynomial coincides with the linear space of invariant directions. Let $L := \{p_2 = 0\} \cap \{x_{n+1} = 0\}$. Then, L is a linear subspace of dimension at most $n-1$ because $p_2 \not\equiv 0$ on the thin space. Now, $p_2(y_\infty) = 0$ by the second part of Lemma 2.4.5, and denoting z_k the orthogonal projections of y_k onto L,

$$\frac{v_k^{(3)}}{\|v_k^{(3)}\|_{L^2(\partial B_1)}} = \frac{p_2(y_k+\cdot) - p_2}{\|p_2(y_k+\cdot) - p_2\|_{L^2(\partial B_1)}} = \frac{p_2(y_k - z_k + \cdot) - p_2}{\|p_2(y_k - z_k+\cdot) - p_2\|_{L^2(\partial B_1)}} \rightharpoonup \nabla p_2 \cdot e,$$

weakly in H^1loc, up to a subsequence, because $y_k - z_k \to 0$, and for some non-zero $e \in L^\perp$.

We now divide the proof into three steps.

Step 1. We prove that

$$\tilde{v}_k := \frac{v_k}{\|v_k\|_{L^2(\partial B_1)}} \rightharpoonup Q \quad \text{in } H^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$$

for some Q with polynomial growth.

By Proposition 2.2.3 and the monotonicity of ϕ, there exist $r_0 > 0$ and $k_0 \in \mathbb{N}$ such that, for $M := \phi(0^+, u(\cdot,0) - p_2) + 1$, we have

$$\phi(r, u(x_k + \cdot, t_k) - p_2) \le M \quad \forall r \in (0, r_0), \quad \forall k \ge k_0$$

and equivalently

$$\phi(r, \tilde{v}_k) = \phi(r, v_k) \le M \quad \forall r \in (0, r_0/r_k), \quad \forall k \ge k_0.$$

Applying Lemma 2.2.9 to v_k, we obtain

$$H(R, \tilde{v}_k) \le C R^{2M+1} H(1, \tilde{v}_k) = C R^{2M+1} \quad \forall R \in [1, r_0/r_k), \quad \forall k \ge k_0,$$

maybe with a smaller $r_0 > 0$, and then $\|\tilde{v}_k\|_{H^1(B_R)} \le C(R)$.

By compactness, it follows that $\tilde{v}_k \rightharpoonup Q$ in $H^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$, up to a subsequence.

Step 2. Observe that q is harmonic by Proposition 2.2.10. We now prove that Q is harmonic as well and grows at most quadratically at the origin.

First, $\Delta\tilde{v}_k \le 0$ in B_{1/r_k}. Moreover, by [65, Proposition 7.1],

$$\|u(x_k + \rho\cdot, t_k) - p_{2,x_k}(\rho\cdot)\|_{L^1(\partial B_1)} \le \rho^2\omega(\rho),$$

with $\omega(\rho) \to 0$ as $\rho \to 0$, and hence

$$\|u(x_k + \rho\cdot, t_k) - p_{2,x_k}(\rho\cdot)\|_{L^\infty(B_1)} \le C\rho^2\omega(\rho).$$

Furthermore, for $R \ge 1$, substituting $\rho = Rr_k \le 1$,

$$\|u(x_k + r_k\cdot, t_k) - p_{2,x_k}(r_k\cdot)\|_{L^\infty(B_R)} \le C(Rr_k)^2\omega(Rr_k),$$

and for any $x \in B_R \cap \{u(x_k + r_k x, t_k) = 0\}$, using that the polynomial is 2-homogeneous,

$$p_{2,x_k}(x) \le CR^2\omega(Rr_k) \Rightarrow p_2(x) \le CR^2\omega(Rr_k),$$

by Lemma 2.4.5.

Then, since p_2 grows quadratically away from its zero set,

$$\begin{aligned} B_R \cap \{u(x_k + r_k\cdot, t_k) = 0\} \cap \{x_{n+1} = 0\} \subset \\ \left\{ y \in B_R : \operatorname{dist}(y, L) \le CR\,[\omega(Rr_k)]^{1/2} \right\} \cap \{x_{n+1} = 0\}, \end{aligned}$$

and the right hand side tends to 0 as $k \to \infty$ for any fixed R. This shows that

$$\sup\{\operatorname{dist}(x, L) : x \in B_R \cap \{u(x_k + r_k\cdot, t_k) = 0\}\} \cap \{x_{n+1} = 0\} \downarrow 0,$$

and it follows that the weak limit of the sequence of nonpositive measures $\Delta\tilde{v}_k$ will be supported on L.

Finally, since L is a linear space of at most dimension $n-1$, given any test function $\xi \in C^\infty_c(\mathbb{R}^{n+1})$, it can be approximated in H^1 norm by $\xi_j \to \xi$ that vanish on L. Hence,

$$\int \nabla Q \cdot \nabla\xi = \lim_{j\to\infty} \int \nabla Q \cdot \nabla\xi_j = -\lim_{j\to\infty} \int \xi_j \Delta Q = 0,$$

and it follows that Q is harmonic. Observe, also, that by Lemma 2.2.9, given that $x_k \in \mathbf{\Gamma}_2$,

$$H(\rho, v_k) \le \rho^4 H(1, v_k) \quad \forall \rho \in (0,1),$$

and hence in the limit $\|Q(\rho\cdot)\|^2_{L^2(\partial B_1)} = H(\rho, Q) \le \rho^4$ for all $\rho \in (0,1)$, so Q is at most quadratic at the origin.

Step 3. We finally prove that $q(y_\infty) = 0$.

First, let $\hat{\varepsilon}_k := \|v_k^{(1)}\|_{L^2(\partial B_1)} + \|v_k^{(2)}\|_{L^2(\partial B_1)} + \|v_k^{(3)}\|_{L^2(\partial B_1)}$ and $\hat{v}_k := v_k/\hat{\varepsilon}_k$. By Step 1 we have $\hat{v}_k \rightharpoonup \hat{Q} = aQ$ for some $a \in [0,1]$. Moreover, by the first observations,

$$v_k^{(2)}/\hat{\varepsilon}_k \rightharpoonup bq(y_\infty + \cdot) := \hat{Q}^{(2)}, \quad v_k^{(3)}/\hat{\varepsilon}_k \rightharpoonup c\nabla p_2 \cdot e := \hat{Q}^{(3)},$$

weakly in H^1loc, for some $b, c \geq 0$.

Then, the following limit is well defined:

$$\hat{Q}^{(1)} := \lim_k v_k^{(1)}/\hat{\varepsilon}_k = \lim_k v_k/\hat{\varepsilon}_k - \lim_k v_k^{(2)}/\hat{\varepsilon}_k - \lim_k v_k^{(3)}/\hat{\varepsilon}_k,$$

and it has a constant sign because all the $v_k^{(1)}$ do. Since $\hat{Q}$, $\hat{Q}^{(2)}$ and $\hat{Q}^{(3)}$ are harmonic, $\hat{Q}^{(1)}$ must be harmonic as well, and by the Liouville theorem, it must be constant. Hence,

$$\hat{Q} = C + bq(y_\infty + \cdot) + c\nabla p_2 \cdot e,$$

and, by the definition of $\hat{\varepsilon}_k$,

$$C\|1\|_{L^2(\partial B_1)} + b\|q(y_\infty + \cdot)\|_{L^2(\partial B_1)} + c\|\nabla p_2 \cdot e\|_{L^2(\partial B_1)} = 1.$$

If $\hat{Q} \equiv 0$, since q is quadratic, we would have $b = 0$. Then, since $\nabla p_2 \cdot e$ is linear, it would follow that all the terms in the sum are zero, a contradiction.

Therefore, $\hat{Q} \not\equiv 0$, i.e. $a \neq 0$. Since Q grows at most quadratically, $b > 0$ and $\nabla Q(0) = 0$. Hence,

$$0 = y_\infty \cdot \nabla \hat{Q}(0) = by_\infty \cdot \nabla q(y_\infty) + cy_\infty \cdot \nabla(\nabla p_2 \cdot e)(0) = 2bq(y_\infty) + 0,$$

where we used that q is 2-homogeneous and $y_\infty \in \{p_2 = 0\}$, and it follows that $q(y_\infty) = 0$, as required. □

Now we are ready to prove our dimensional bound on $\mathbf{\Gamma}_2^{\mathrm{a}}$.

Proof of Proposition 2.4.3. We need to prove that, for any $\beta > n - 2$, the set $\mathbf{\Gamma}_2^{\mathrm{a}}$ has zero β-dimensional Hausdorff measure. Assume by contradiction that

$$\mathcal{H}^\beta(\mathbf{\Gamma}_2^{\mathrm{a}}) > 0.$$

Then, by the basic properties of Hausdorff measures (see [89, 2.10.19(2)]) there exists a point $x_0 \in \mathbf{\Gamma}_2^{\mathrm{a}}$ (let us assume $x_0 = 0$ without loss of generality), a sequence $r_k \downarrow 0$ and a set $A \subset \overline{B_1}$, with $\mathcal{H}^\beta(A) > 0$, such that for every point $y \in A$, there is a sequence $x_k \in \mathbf{\Gamma}_2^{\mathrm{a}}$ such that $x_k/r_k \to y$.

Let $w = u(\cdot, 0) - p_2$, $w_r = w(r\cdot)$ and $\tilde{w}_r = w_r/H(1, w_r)^{1/2}$. Then, by assumption,

$$\tilde{w}_{r_k} \rightharpoonup q \text{ in } H^1\text{loc}$$

up to a subsequence, where q is a 2-homogeneous harmonic polynomial.

Furthermore, by Lemma 2.4.6 we have $A \subset \{q = 0\} \cap \{p_2 = 0\} \cap \{x_{n+1} = 0\}$. Then, since $\mathcal{H}^\beta(A) > 0$, with $\beta > n - 2$, the only possibility is that $\dim(\{p_2 = 0\} \cap \{x_{n+1} = 0\}) = n - 1$, and that $q \equiv 0$ on $\{p_2 = 0\} \cap \{x_{n+1} = 0\}$. Hence, after a change of variables, we may assume $p_2(x', 0) = x_1^2$, and therefore $p_2(x) = x_1^2 - x_{n+1}^2$, and $q(x) = x_1(a \cdot x) - a_1 x_{n+1}^2$.

Now, by the first part of Lemma 2.2.11,

$$0 = \int_{\partial B_1} (x_1^2 - x_{n+1}^2)(x_1(a \cdot x) - a_1 x_{n+1}^2) = a_1 \int_{\partial B_1} (x_1^2 - x_{n+1}^2)^2,$$

where we used that, for $i > 1$, $x_1 x_i$ is odd with respect to x_1 and $x_1^2 - x_{n+1}^2$ is even. It follows that $a_1 = 0$.

On the other hand, using the second part of Lemma 2.2.11, and letting $p = C(x_1^2 + x_i^2 - 2x_{n+1}^2) + a_i x_1 x_i$ with $i > 1$, and $C > 0$ large enough such that $p(x', 0) \geq 0$,

$$0 \geq \int_{\partial B_1} (C(x_1^2 + x_i^2 - 2x_{n+1}^2) + a_i x_1 x_i)(x_1(a \cdot x)) = a_i^2 \int_{\partial B_1} x_1^2 x_i^2,$$

using again the odd and even symmetries of the terms involved. We conclude that $a_i = 0$ for all $i = 2, \ldots, n$. But then it follows that $q \equiv 0$, a contradiction. □

2.5 Cubic points

In this section, we improve the cleaning rate of the cubic points using a barrier argument combining [102, Lemma 9.4] with Theorem 2.2.12 and the Hopf-type estimate in Lemma 2.2.2.

Proposition 2.5.1. *Let $u : B_1 \times [-1, 1] \to \mathbb{R}$ be a solution to (2.2)-(2.4), with $0 \in \Gamma_3(u(\cdot, 0))$. Then, there exist some $r_0, c_0 > 0$ such that, for all $t \in (-1, 0]$,*

$$\{x \in B_{r_0} : |x|^{2+\gamma} < -c_0 t\} \cap \Gamma(u(\cdot, t)) = \emptyset,$$

for some $\gamma > 0$ only depending on n.

Proof. Let $c_0, \gamma > 0$ to be chosen later. We will prove that there exists $0 < r_0 < \frac{1}{8}$ such that for all $r \in (0, r_0)$, and t with $-c_0 t \geq r^{2+\gamma}$,

$$u(\cdot, t) \equiv 0 \text{ on } B_r \cap \{x_{n+1} = 0\},$$

and in particular there are no free boundary points there.

By Theorem 2.2.12 and Lemma 2.2.13,

$$\|r^{-3} u(r\cdot, 0) - p_3\|_{L^\infty(B_2)} \leq C r^\alpha, \quad p_3(x', x_{n+1}) = |x_{n+1}|(a x_{n+1}^2 - x' \cdot A x'),$$

with $a \geq 0$ and A symmetric and nonnegative definite.

Let us then bound $v(x) := r^{-3} u(rx, t)$. By Lemma 2.2.2 (after reversing t) and the previous estimates,

$$v(x) \leq r^{-3} u(rx, 0) - c r^{-3} |t| |r x_{n+1}| \leq a |x_{n+1}|^3 + C r^\alpha - C_1 r^\gamma |x_{n+1}| \text{ in } B_2,$$

where $C_1 = c/c_0$. Now, given $z' \in \mathbb{R}^n$ with $|z'| < 1$, and $\delta \geq 0$, we introduce the barrier

$$\psi_{z',\delta}(x', x_{n+1}) = -(n+1) x_{n+1}^2 + (x' - z')^2 + \delta.$$

Let $z = (z', 0)$, and let $s = (C r^\alpha)^{1/2}$, which is smaller than 1 for sufficiently small r. We will prove that $v \leq \psi_{z',\delta}$ in $B_s(z)$. First, given $x \in \partial B_s(z)$, using that $(x' - z')^2 = s^2 - x_{n+1}^2$, it suffices to see that

$$a |x_{n+1}|^3 + C r^\alpha - C_1 r^\gamma |x_{n+1}| \leq -(n+2) x_{n+1}^2 + s^2 \text{ for } |x_{n+1}| \leq s,$$

which after choosing $s = (C r^\alpha)^{1/2}$ becomes

$$C_1 r^\gamma |x_{n+1}| \geq a |x_{n+1}|^3 + (n+2) x_{n+1}^2 \text{ for } |x_{n+1}| \leq (C r^\alpha)^{1/2},$$

that is satisfied choosing $\gamma = \alpha/2$ and a sufficiently large C_1 (i.e., a sufficiently small c_0).

Let us assume that there exists $\delta > 0$ such that $\psi_{z',\delta}$ touches v from above in $\overline{B}_s(z)$ at x_0. Observe that $x_0 \in B_s(z)$ because $\psi_{z',\delta} > v$ on $\partial B_s(z)$ for all positive δ. Now, if $x_0 \notin \{x_{n+1} = 0, v = 0\}$, $\Delta v(x_0) = 0$ and $\Delta \psi_{z',\delta} = -2$, we have a superharmonic function touching a harmonic function from above, which is a contradiction. On the other hand, if x_0 belongs to the contact set,

$$0 = v(x_0) = \psi_{z',\delta}(x_0) = (x_0' - z')^2 + \delta > 0,$$

a contradiction as well. Therefore, the only possibility is that $v \le \psi_{z',\delta}$ in $B_s(z)$ for all $\delta > 0$, and in particular $v(z) \le 0$.

Repeating the argument for all $z \in B_1 \cap \{x_{n+1} = 0\}$, we obtain that $v \equiv 0$ on $B_1 \cap \{x_{n+1} = 0\}$, which is the same as $u(\cdot, t) \equiv 0$ on $B_r \cap \{x_{n+1} = 0\}$. □

2.6 Proof of Theorem 2.1.2

We take advantage of the following stratification of the degenerate set to compute our estimates:

$$\mathrm{Deg} = \mathbf{\Gamma}_2^{\mathrm{o}} \cup \mathbf{\Gamma}_2^{\mathrm{a}} \cup \mathbf{\Gamma}_3 \cup \mathbf{\Gamma}_{\ge 7/2} \cup \mathbf{\Gamma}_*.$$

We can now apply Proposition 2.2.15 to obtain generic dimensional estimates for all of these sets.

Proposition 2.6.1. *Let $u : B_1 \times [-1, 1] \to \mathbb{R}$ be a solution to (2.2)-(2.4). Let $\pi_2 : (x, t) \mapsto t$ be the standard projection. Then, there exist $\alpha, \gamma > 0$, depending only on n, such that:*

(a) If $n = 1$,

- $\mathbf{\Gamma}_2^{\mathrm{o}}$ *is discrete,*
- $\mathbf{\Gamma}_2^{\mathrm{a}} = \emptyset$,
- $\mathbf{\Gamma}_3 = \emptyset$,
- $\mathbf{\Gamma}_{\ge 7/2}$ *is discrete,*
- $\mathbf{\Gamma}_* = \emptyset$.

(b) If $n = 2$,

- $\dim_{\mathcal{H}}(\pi_2(\mathbf{\Gamma}_2^{\mathrm{o}})) \le 1/3$,
- $\mathbf{\Gamma}_2^{\mathrm{a}}$ *is discrete,*
- $\dim_{\mathcal{H}}(\pi_2(\mathbf{\Gamma}_3)) \le 1/(2+\gamma)$,
- $\dim_{\mathcal{H}}(\pi_2(\mathbf{\Gamma}_{\ge 7/2})) \le 2/5$,
- $\mathbf{\Gamma}_*$ *is discrete.*

(c) If $n = 3$,

- $\dim_{\mathcal{H}}(\pi_2(\mathbf{\Gamma}_2^{\mathrm{o}})) \le 2/3$,
- $\dim_{\mathcal{H}}(\pi_2(\mathbf{\Gamma}_2^{\mathrm{a}})) \le 1/2$,

- $\dim_{\mathcal{H}}(\pi_2(\mathbf{\Gamma}_3)) \leq 2/(2+\gamma)$,
- $\dim_{\mathcal{H}}(\pi_2(\mathbf{\Gamma}_{\geq 7/2})) \leq 4/5$,
- $\dim_{\mathcal{H}}(\pi_2(\mathbf{\Gamma}_*)) \leq 1/(1+\alpha)$.

(d) If $n \geq 4$, for $\mathcal{H}^1$-a.e. $t \in [-1,1]$,

- $\dim_{\mathcal{H}}(\Gamma_2^{\mathrm{o}}(u(\cdot,t))) \leq n-4$,
- $\dim_{\mathcal{H}}(\Gamma_2^{\mathrm{a}}(u(\cdot,t))) \leq n-4$,
- $\dim_{\mathcal{H}}(\Gamma_3(u(\cdot,t))) \leq n-3-\gamma$,
- $\dim_{\mathcal{H}}(\Gamma_{\geq 7/2}(u(\cdot,t))) \leq n-\frac{7}{2}$,
- $\dim_{\mathcal{H}}(\Gamma_*(u(\cdot,t))) \leq n-3-\alpha$.

Proof. For each of the sets considered, we combine a total dimension estimate with a *cleaning* result.

- For $\mathbf{\Gamma}_2^{\mathrm{o}}$, by Proposition 2.3.1(a), $\dim_{\mathcal{H}}(\mathbf{\Gamma}_2^{\mathrm{o}}) \leq n-1$, and $\mathbf{\Gamma}_2^{\mathrm{o}}$ is discrete when $n=1$. By Proposition 2.4.1, for all $x_0 \in \mathbf{\Gamma}_2^{\mathrm{o}}$ and for all $\varepsilon > 0$, there exist $r_0, c > 0$ such that
$$\{x \in B_{r_0} : |x-x_0|^{3-\varepsilon} < (t-\tau(x_0))\} \cap \mathbf{\Gamma}_2^{\mathrm{o}} = \emptyset.$$
- For $\mathbf{\Gamma}_2^{\mathrm{a}}$, by Proposition 2.4.3, $\dim_{\mathcal{H}}(\mathbf{\Gamma}_2^{\mathrm{a}}) \leq n-2$, $\mathbf{\Gamma}_2^{\mathrm{a}}$ is discrete when $n=2$, and it is empty when $n=1$. By Proposition 2.2.7, for all $x_0 \in \mathbf{\Gamma}_2$ and for all $\varepsilon > 0$, there exist $r_0, c > 0$ such that
$$\{x \in B_{r_0} : |x-x_0|^{2-\varepsilon} < (t-\tau(x_0))\} \cap \mathbf{\Gamma}_2 = \emptyset.$$
- For $\mathbf{\Gamma}_3$, by Proposition 2.3.1(a), $\dim_{\mathcal{H}}(\mathbf{\Gamma}_3) \leq n-1$, and $\mathbf{\Gamma}_3$ is discrete when $n=1$. By Proposition 2.5.1, for all $x_0 \in \mathbf{\Gamma}_3$, there exist $r_0, c > 0$ such that
$$\{x \in B_{r_0} : |x-x_0|^{2+\gamma} < -c(t-\tau(x_0))\} \cap \mathbf{\Gamma}_3 = \emptyset,$$
and after changing t by $-t$, for all $\varepsilon > 0$ there exists $r_1 > 0$ such that for all $r \in (0, r_1)$,
$$B_r(x_0) \cap \{(x,t) : x \in \Gamma_3(u(\cdot,t))\} = \emptyset$$
for all $t > \tau(x_0) + c^{-1}r^{2+\gamma} \geq \tau(x_0) + r^{2+\gamma-\varepsilon}$.
- For the set $\mathbf{\Gamma}_{\geq 7/2}$, by Proposition 2.3.1(a), $\dim_{\mathcal{H}}(\mathbf{\Gamma}_{\geq 7/2}) \leq n-1$, and $\mathbf{\Gamma}_{\geq 7/2}$ is discrete when $n=1$. By Proposition 2.2.7, for all $x_0 \in \mathbf{\Gamma}_{\geq 7/2}$ and for all $\varepsilon > 0$, there exists $r_0 > 0$ such that
$$\{x \in B_{r_0} : |x-x_0|^{5/2-\varepsilon} < (t-\tau(x_0))\} \cap \mathbf{\Gamma}_{\geq 7/2} = \emptyset.$$
- Finally, for $\mathbf{\Gamma}_*$, by Proposition 2.3.1(b), $\dim_{\mathcal{H}}(\mathbf{\Gamma}_*) \leq n-2$, $\mathbf{\Gamma}_*$ is discrete when $n=2$, and it is empty when $n=1$. Then, thanks to [65, Theorem 4], the order of the points in $\mathbf{\Gamma}_*$ is $\kappa \geq 2+\alpha$ for some dimensional $\alpha > 0$. Applying Proposition 2.2.7 as in the previous case, for all $x_0 \in \mathbf{\Gamma}_*$ and for all $\varepsilon > 0$, there exists $r_0 > 0$ such that
$$\{x \in B_{r_0} : |x-x_0|^{1+\alpha-\varepsilon} < (t-\tau(x_0))\} \cap \mathbf{\Gamma}_* = \emptyset.$$

The conclusions follow now by Proposition 2.2.15. □

Finally, we can prove our main results.

Proof of Theorem 2.1.2. It is a direct consequence of Proposition 2.6.1. □

Proof of Conjecture 2.1.1. It is a direct consequence of Proposition 2.6.1. The smoothness of the free boundary follows from [133, 72]. □

Chapter 3

Semiconvexity estimates for nonlinear integro-differential equations

In this paper we establish for the first time local semiconvexity estimates for fully nonlinear equations and for obstacle problems driven by integro-differential operators with general kernels. Our proof is based on the Bernstein technique, which we develop for a natural class of nonlocal operators and consider to be of independent interest. In particular, we solve an open problem from Cabré-Dipierro-Valdinoci [33]. As an application of our result, we establish optimal regularity estimates and smoothness of the free boundary near regular points for the nonlocal obstacle problem on domains. Finally, we also extend the Bernstein technique to parabolic equations and nonsymmetric operators.

3.1 Introduction

The aim of this work is to establish semiconvexity estimates for solutions to nonlinear equations driven by integro-differential operators of the form

$$Lu(x) = \text{p.v.} \int_{\mathbb{R}^n} \Big(u(x) - u(y)\Big) K(x-y) \, \mathrm{d}y, \tag{3.1}$$

where $K : \mathbb{R}^n \to [0, \infty]$ is comparable to the kernel of the fractional Laplacian. To be precise, we consider the following natural class of symmetric jumping kernels K (see [52], [51]) satisfying the classical uniform ellipticity condition

$$\lambda |y|^{-n-2s} \le K(y) \le \Lambda |y|^{-n-2s} \tag{$K_\asymp$}$$

for some constants $0 < \lambda \le \Lambda$ and $s \in (0,1)$, and the smoothness conditions

$$|\nabla K(y)| \le \Lambda |y|^{-1} K(y), \tag{C^1}$$

$$|D^2 K(y)| \le \Lambda |y|^{-2} K(y). \tag{C^2}$$

We denote the family of all such operators by $\mathcal{L}_s(\lambda, \Lambda; 2)$; see also Definition 3.2.1.

3.1.1 Fully nonlinear equations

The regularity theory for fully nonlinear nonlocal equations was developed by Caffarelli and Silvestre in their celebrated series of papers [50], [51], and [52]. They established a nonlocal counterpart of the Krylov-Safonov theorem, stating that solutions are $C^{1+\varepsilon}$, and an Evans-Krylov theorem, which yields $C^{2s+\varepsilon}$-regularity of solutions to concave equations. Let us also refer to [54], [116], [137], [181], [180], [56], [179], [124], [57], [80] for extensions of the aforementioned results, e.g., to operators with coefficients and to parabolic problems; see also [98] and [16, 17, 18].

It remains an intriguing open problem after [51] to establish higher regularity of solutions to nonlocal Bellman-type equations. So far, it is still unknown whether solutions are more regular than $C^{1+\varepsilon} \cap C^{2s+\varepsilon}$, even if the underlying class of operators possesses only smooth kernels.

Here we prove for the first time semiconvexity estimates for solutions to these equations:

Theorem 3.1.1. *Let $s \in (0,1)$, and let u be any viscosity solution to a fully nonlinear equation*

$$\inf_{\gamma \in \Gamma} \{L_\gamma u\} = 0 \quad \text{in } B_1, \tag{3.2}$$

where $\{L_\gamma\}_{\gamma \in \Gamma} \subset \mathcal{L}_s(\lambda, \Lambda; 2)$. Then, u satisfies

$$\partial^2_{ee} u \geq -C\|u\|_{L^\infty(\mathbb{R}^n)} \quad \text{in } B_{1/2}$$

for all $e \in \mathbb{S}^{n-1}$, where C depends only on n, s, λ, Λ.

Our Theorem 3.1.1 establishes one-sided $C^{1,1}$-regularity estimates and can be seen as the first contribution after [51] to the higher regularity for solutions to (3.2). Note that the class of operators $\mathcal{L}_s(\lambda, \Lambda; 2)$ is also considered in [51]. A more general version of the result, where we allow for an x-dependent right hand side, will be proved in Theorem 3.4.2.

Semiconvexity estimates play a crucial role in the study of nonlinear elliptic PDE. For instance, semiconvexity estimates for solutions to second order fully nonlinear PDE imply two-sided $C^{1,1}$ regularity estimates, if the operator under consideration is uniformly elliptic (see [37]). In light of this, it is an interesting question to ask whether Theorem 3.1.1 also implies $C^{1,1}$-regularity estimates for (3.2)[1], at least when $s \geq 1/2$.

Let us emphasize that, for nonlocal fully nonlinear equations, the only known semiconvexity estimate was that of Cabré, Dipierro, and Valdinoci [33], who proved (a priori) semiconvexity estimates for fully nonlinear equations built from affine transformations of the fractional Laplacian.

3.1.2 Obstacle problems

Semiconvexity estimates play a crucial role in the study of obstacle problems

$$\min\{Lu, u - \phi\} = 0 \quad \text{in } \Omega \subset \mathbb{R}^n.$$

While the classical obstacle problem (corresponding to $L = -\Delta$) is very well understood [34, 160, 97], the case of integro-differential operators L is significantly more complicated,

[1] Even in the local case, optimal regularity for Bellman equations remains an open problem; see e.g. [97, Chapter 4.5]. The only partial result in this direction is due to Caffarelli, De Silva, and Savin [40], who established optimal $C^{2,1}$ regularity in case $n = 2$ and for equations of the type $\min\{L_1 u, L_2 u\} = 0$.

and several questions remain open. The regularity theory for nonlocal obstacle problems was initiated in the seminal works [11, 185, 48] for $L = (-\Delta)^s$, and further developed in [110, 159, 133, 123, 111, 107, 95, 65, 64, 93, 139, 175]. In that case, one can identify the obstacle problem for $(-\Delta)^s$ with a (weighted) thin obstacle problem in $\mathbb{R}^{n+1}_+$ thanks to the celebrated Caffarelli-Silvestre extension [49], and this gives access to many local techniques. In particular, it allows to prove local semiconvexity estimates in the extended variables; see [7, 11] for the case $s = \frac{1}{2}$, and [33, 93] for all $s \in (0, 1)$.

The regularity theory for obstacle problems (3.1.2) with general integro-differential operators L requires quite different methods compared to the case $L = (-\Delta)^s$, and has been developed in [46, 39, 1, 104]. The best known results so far establish the optimal C^{1+s}-regularity of solutions [104], as well as the smoothness of free boundaries near regular points [46, 1], whenever solutions are *semiconvex*. The semiconvexity property holds true for any global solution[2] (i.e. when $\Omega = \mathbb{R}^n$), which follows by a simple translation argument based on the maximum principle [185, 46].

An important open problem was to establish *local* regularity estimates for solutions of (3.1.2), not relying on any a priori assumptions on the boundary data. We solve this by proving *local* semiconvexity estimates, which is exactly the content of our next result.

Theorem 3.1.2. *Let $s \in (0, 1)$, $L \in \mathcal{L}_s(\lambda, \Lambda; 2)$, and u be any solution to the nonlocal obstacle problem*

$$\min\{Lu, u - \phi\} = 0 \quad \text{in } B_1, \tag{3.3}$$

where $\phi \in C^4(\mathbb{R}^n)$. Then, u satisfies

$$\partial^2_{ee} u \geq -C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|L\phi\|_{C^{1,1}(B_1)}) \quad \text{in } B_{1/2}$$

for all $e \in \mathbb{S}^{n-1}$, where C depends only on n, s, λ, Λ.

As said above, semiconvexity estimates are a key tool in the regularity theory for nonlocal obstacle problems, mainly because they imply the convexity of blow-ups. Without this, one cannot establish the optimal regularity of solutions and the regularity of free boundaries. Moreover, as we will see next, this also allows us to study obstacle problems under minimal regularity assumptions on the obstacle ϕ (which is new even in the global case $\Omega = \mathbb{R}^n$).

Regularity of solutions and free boundaries

Let us explain now the main applications of our new semiconvexity estimate for nonlocal obstacle problems. We consider general nonlocal operators belonging to the class $\mathcal{L}_s(\lambda, \Lambda; 1)$, i.e., having symmetric jumping kernels K satisfying[3] the ellipticity condition ($K_\asymp$) and (C^1). As in [46, 1, 104], we also need to assume that K is homogeneous, i.e.,

$$K(y) = \frac{K(y/|y|)}{|y|^{n+2s}} \qquad \text{for all} \quad y \in \mathbb{R}^n \setminus \{0\}. \tag{3.4}$$

Our first result in this direction is to obtain for the first time *local* C^{1+s}-estimates.

[2]Also, if one assumes that the contact set $\{u = \phi\}$ is compactly contained in Ω, then one can use a cutoff argument to transform the problem into a global one. However, this does not give local regularity estimates, and all constants would depend on the distance from $\{u = \phi\}$ to $\partial\Omega$.

[3]Note that, in contrast to global solutions, optimal regularity for local solutions to nonlocal obstacle problems cannot hold true in general without some regularity assumption on K, or prescribing regularity on the complement data (see [166, Proposition 6.1], [180, Section 5.1]).

Corollary 3.1.3 (Optimal regularity)**.** *Let $s \in (0,1)$ and $L \in \mathcal{L}_s(\lambda,\Lambda;1)$ satisfying (3.4). Let u be any solution to the obstacle problem*

$$\min\{Lu, u-\phi\} = 0 \quad \text{in } B_1,$$

with $\phi \in C^\beta(B_1)$ for some $\beta > 0$. Then,

$$\|u\|_{C^\beta(B_{1/2})} \le C(\|\phi\|_{C^\beta(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}) \qquad \text{for} \quad \beta < 1+s,$$

while

$$\|u\|_{C^{1+s}(B_1)} \le C(\|\phi\|_{C^\beta(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}) \qquad \text{for} \quad \beta > 1+2s.$$

The constant C depends only on n, s, β, λ, and Λ.

The key point of this result is that the estimates are completely local, i.e., they do not depend on any assumption on the boundary data nor on the obstacle on ∂B_1. Still, even in case of global solutions, observe that Theorem 3.1.3 only requires[4] obstacles $\phi \in C^\beta$ for some $\beta > 1+2s$ in order to obtain optimal C^{1+s}-regularity estimates for solutions, which even improves the results in [46, 104] for the global nonlocal obstacle problem in case $s < \frac{1}{2}$. Furthermore, the optimal C^β-regularity in case of non-smooth obstacles for $\beta < 1+s$ is also new even in case of global solutions; it was only known for $\beta \le \max\{1+\varepsilon, 2s+\varepsilon\}$ [39, Theorem 5.1].

Our second result in this direction establishes the regularity of the free boundary near regular points for local solutions to (3.1.3).

Corollary 3.1.4 (Free boundary regularity)**.** *Let $s \in (0,1)$ and $L \in \mathcal{L}_s(\lambda,\Lambda;1)$ satisfying (3.4). Let u be any solution to the obstacle problem (3.1.3) with $\phi \in C^\beta(B_1)$ for some $\beta > 1+2s$, and let $\alpha \in (0,s) \cap (0,1-s)$. Then, near any free boundary point $x_0 \in \partial\{u > \phi\} \cap B_1$, there exist $c_0 \ge 0$ and $e \in \mathbb{S}^{n-1}$ such that for any $x \in B_1(x_0)$:*

$$\left| u(x) - \phi(x) - c_0\left((x-x_0)\cdot e\right)_+^{1+s} \right| \le C(\|\phi\|_{C^\beta(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)})|x-x_0|^{1+s+\alpha},$$

where C depends only on n, s, λ, Λ, and α.
Moreover, if $c_0 > 0$, then the free boundary is a $C^{1,\alpha}$-graph in a ball $B_{\rho_0}(x_0)$ with $C\rho_0^\alpha \ge c_0$ and C depending only on n, s, λ, Λ, and α.

If we assume in addition that $\phi \in C^\infty(B_1)$ and $K|_{\mathbb{S}^{n-1}} \in C^\infty(\mathbb{S}^{n-1})$, then the free boundary is actually C^∞ near regular points (i.e., near those points x_0 for which $c_0 > 0$); see [1].

Convexity of blow-ups

As explained before, the semiconvexity of solutions to nonlocal obstacle problems is crucial in the study of the regularity of the free boundary and of optimal regularity estimates. The approach in [11, 185, 48, 46, 104] is to show first that solutions are semiconvex and to prove that blow-ups, as limits of correctly rescaled semiconvex solutions, are then convex. The analysis of the free boundary heavily relies on the classification of blow-ups, and their convexity is of central importance in this procedure; see [46, 48].

Here, we are able to establish the convexity of the blow-ups directly, as follows.

[4] In case $L = (-\Delta)^s$, it was proved in [39, Theorem 6.1] that optimal C^{1+s} estimates already hold when $\phi \in C^\beta$ with $\beta > 1+s$. Their proof is based on a truncated version of Almgren's monotonicity formula for the extension. Due to the lack of monotonicity formulas in the more general setting of nonlocal operators $L \in \mathcal{L}_s(\lambda,\Lambda;1)$, such proof cannot work in our context. Still, we expect that the same result for $\beta > 1+s$ can be established for general kernels using our new result on the convexity of blow-ups, Theorem 3.1.5.

Theorem 3.1.5 (Convexity of blow-ups)**.** *Let $s \in (0,1)$, $L \in \mathcal{L}_s(\lambda, \Lambda; 1)$, and $\alpha \in (0,s) \cap (0, 1-s)$. Let $u_0 \in C^{0,1}_{loc}(\mathbb{R}^n) \cap C^{2s+\varepsilon}_{loc}(\mathbb{R}^n)$ be such that $u_0 \geq 0$ in $\mathbb{R}^n$,*

$$\|\nabla u_0\|_{L^\infty(B_R)} \leq cR^{s+\alpha} \qquad \text{for all} \quad R \geq 1, \tag{3.5}$$

$$L\left(\frac{u_0(x+h) - u_0(x)}{|h|}\right) \geq 0 \qquad \text{for all} \quad x \in \{u_0 > 0\}, \quad h \in \mathbb{R}^n. \tag{3.6}$$

Then, u_0 is convex, i.e., $D^2 u_0 \geq 0$ in $\mathbb{R}^n$.

In particular, if in addition (3.4) holds and $u \in C^1(\{u_0 > 0\})$ solves in the viscosity sense $L(\nabla u_0) = 0$ in $\{u_0 > 0\}$, then $u_0(x) = \kappa(x \cdot \nu)_+^{1+s}$, with $\nu \in \mathbb{S}^{n-1}$, $\kappa \geq 0$.

A key advantage of establishing the convexity of blow-ups without relying on semiconvexity of solutions is that we can relax the regularity assumptions of the obstacle ϕ in order to obtain optimal regularity estimates, as in Theorem 3.1.3 above. Moreover, it also opens the door to proving regularity estimates for x-dependent operators; we plan to study this in future works.

3.1.3 Bernstein technique for nonlocal operators

A central contribution of this article is the development of the Bernstein technique to obtain semiconvexity estimates for general integro-differential operators, solving an important open problem left in [33]. This technique plays an important role in the proofs of our main results concerning fully nonlinear equations and obstacle problems (see Theorem 3.1.1 and Theorem 3.1.2 above). Still, we emphasize that, as in the local case, our nonlocal Bernstein technique works in a rather general framework, so it could be useful in completely different contexts (see e.g. [63] and [5] for applications of the classical Bernstein technique to Hamilton-Jacobi equations and double obstacle problems).

The main insight behind the Bernstein technique is that, if derivatives of the solution u are also subsolutions to an equation, then the maximum principle can be used in order to obtain regularity estimates for these solutions. This observation can be traced back to Serge Bernstein (see [25, 26]), who noticed that

$$u \text{ harmonic in } B_1 \quad \Rightarrow \quad |\nabla u|^2 \text{ subharmonic in } B_1.$$

By the maximum principle, this yields an estimate of $\|\nabla u\|_{L^\infty(B_1)}$ by $\|\nabla u\|_{L^\infty(\partial B_1)}$. This idea can be generalized by considering, instead of $|\nabla u|^2$, the following auxiliary function

$$\psi = \eta^2(\partial_e u)^2 + \sigma u^2, \tag{3.7}$$

which allows to prove interior regularity estimates for $\partial_e u$ by lower order terms. Here, $\eta \in C^\infty_c(\mathbb{R}^n)$ is a cut-off function, and $\sigma > 0$ is a suitably chosen constant. Such choice of ψ has already been applied in [37] in the context of (local) fully nonlinear PDEs. Different auxiliary functions appear for instance in the context of the mean curvature equation (see [182], [200]). Moreover, we refer to [142], [82], [15], [112], and [7], [90], [93], [33], [5] for further references on the Bernstein technique applied to elliptic equations of second order and to free boundary problems, respectively.

So far, the only work on the Bernstein technique for integro-differential operators is [33], where the authors consider the auxiliary function (3.7) and are able to treat the family of nonlocal operators that arise as affine transformations of the fractional Laplacian (i.e., kernels of the form $K(y) = |Ay|^{-n-2s}$ for some symmetric, uniformly elliptic matrix A). Moreover, they obtain Lipschitz estimates for operators belonging to the class $\mathcal{L}_s(\lambda, \Lambda; 2)$ — but not semiconvexity estimates like the ones in Theorems 3.1.1 and 3.1.2 above. A central contribution of our work is to extend and improve the results of [33] by establishing the Bernstein technique for general nonlocal operators belonging to the natural class $\mathcal{L}_s(\lambda, \Lambda; 1)$, and to solve all the problems that were left open after [33]. Moreover, as explained in more detail in Section 3.7, we are able to extend our method to treat

- parabolic equations,
- nonsymmetric operators with drifts, and
- nonlocal operators not necessarily comparable to $(-\Delta)^s$.

Key estimates

The main ingredient in the proof of our semiconvexity estimates via the Bernstein technique is the following:

Theorem 3.1.6. *Let $s \in (s_0, 1)$, with $s_0 > 0$, and $L \in \mathcal{L}_s(\lambda, \Lambda; 1)$. Let $\eta \in C^{1,1}(\mathbb{R}^n)$ be such that $\eta \geq 0$. Then, there exists $\sigma_0 = \sigma_0(n, s, \Lambda/\lambda, \|\eta\|_{C^{1,1}(\mathbb{R}^n)}) > 0$ such that for every $\sigma \geq \sigma_0$ and every $u \in C^{1+2s+\varepsilon}_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*

$$L\Big(\eta^2 (\partial_e u)^2 + \sigma u^2\Big) \leq 2\eta^2 L(\partial_e u)\partial_e u + 2\sigma L(u) u \qquad in \quad \mathbb{R}^n. \tag{3.8}$$

We emphasize that Theorem 3.1.6 has so far only been established in [33] for affine transformations of the fractional Laplacian, by using the Caffarelli-Silvestre extension. Finding a proof of (3.8) without using the extension remained open after [33], even in case $L = (-\Delta)^s$ (see Open problem 1.6 in [33]). As we explain below, our proof merely relies on elementary arguments using the precise shape of L (see (3.1)), thereby solving Open problem 1.6 in [33]. Moreover, our approach allows us to obtain Theorem 3.1.6 for the natural class of operators $\mathcal{L}_s(\lambda, \Lambda; 1)$ without any additional remainder terms as in [33]. This solves Open problem 1.7 in [33].

It is important to emphasize that, for linear operators of second order, (3.8) is a simple consequence of the product rule. However, for nonlocal operators such estimate is very far from trivial, and was left as an open problem in [33].

Remark 3.1.7. The key estimate Theorem 3.1.6 is robust with respect to the limit $s \nearrow 1$, i.e., the constants σ_0 and C depend only on n, s_0, λ, Λ. Since any operator $\mathcal{L}$ of the form

$$\mathcal{L}u(x) = \sum_{i,j=1}^n a_{ij}\partial_{ij}u(x), \qquad \text{with} \quad \lambda|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \ \ \forall \xi \in \mathbb{R}^n \tag{3.9}$$

can be approximated by a sequence of operators $L_s \in \mathcal{L}_s((1-s)\lambda, (1-s)\Lambda; 1)$ as $s \nearrow 1$, our results are a true generalization of the corresponding ones for operators (3.9).

Observe that the key estimate Theorem 3.1.6 is not suitable for proving one-sided regularity estimates, such as Theorem 3.1.1 and Theorem 3.1.2. Therefore, in order to establish semiconvexity estimates, we rely on one-sided key estimates of the following form:

Theorem 3.1.8. *Let $s \in (s_0, 1)$, with $s_0 > 0$, and $L \in \mathcal{L}_s(\lambda, \Lambda; 1)$. Let $\eta \in C^{1,1}(\mathbb{R}^n)$ be such that $\eta \ge 0$. Then, there exists $\sigma_0 = \sigma_0(n, s, \Lambda/\lambda, \|\eta\|_{C^{1,1}(\mathbb{R}^n)}) > 0$ such that for every $\sigma \ge \sigma_0$ and every $v \in C^{1+2s+\varepsilon}_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*

$$L\Big(\eta^2(\partial_e v)_+^2 + \sigma v^2\Big) \le 2\eta^2 L(\partial_e v)(\partial_e v)_+ + 2\sigma L(v)v \qquad \textit{in} \quad \mathbb{R}^n. \tag{3.10}$$

We will prove the one-sided estimate (3.10) by a slight modification of the proof of Theorem 3.1.6. Before our work, (3.10) was only known for affine transformations of the fractional Laplacian (see [33]). Therefore, after [33], it even remained unclear, whether semiconvexity estimates such as Theorem 3.1.1 for nonlocal fully nonlinear equations driven by operators from the class $\mathcal{L}_s(\lambda, \Lambda; 1)$ do hold true (see Open problem 1.8 in [33]). Theorem 3.1.8 is the main ingredient in our proofs of the desired semiconvexity estimates Theorem 3.1.1, Theorem 3.1.2, the first of which solves Open problem 1.8 in [33].

Difference quotients

In (3.8), it is apparent that one needs to consider sufficiently smooth solutions u in order for $L(\partial_e u)$ to be well-defined ($u \in C^{1+2s+\varepsilon}_{loc}(\mathbb{R}^n)$ is sufficient, for any $\varepsilon > 0$). In case of fully nonlinear equations, this is not a problem, as one can approximate any viscosity solution u by smooth solutions u^ε, thanks to the results in [92]. However, in the obstacle problem, solutions are never more regular than C^{1+s}, and therefore, Theorem 3.1.6 and Theorem 3.1.8 cannot be used directly to derive semiconvexity estimates. In order to be able to apply the Bernstein technique to the obstacle problem (and also to other equations where such approximation argument is not possible), we prove an estimate reminiscent of (3.8) for difference quotients. To this end, we introduce for $h \in \mathbb{R}^n$

$$u_h(x) := \int_0^1 u(x + th)\,\mathrm{d}t \qquad \text{and} \qquad D_h u(x) = \frac{u(x+h) - u(x)}{|h|},$$

and establish the following.

Proposition 3.1.9. *Let $s \in (s_0, 1)$, with $s_0 > 0$, and $L \in \mathcal{L}_s(\lambda, \Lambda; 1)$. Let $\eta \in C^{1,1}(\mathbb{R}^n)$ be such that $\eta \ge 0$, and $|h| \le 1/8$. Then, there exists $\sigma_0 = \sigma_0(n, s, \Lambda/\lambda, \|\eta\|_{C^{1,1}(\mathbb{R}^n)}) > 0$ such that for every $\sigma \ge \sigma_0$ and every $u, v \in C^{2s+\varepsilon}_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*

$$L(\eta^2(D_h u)^2 + \sigma u_h^2) \le 2\eta^2 L(D_h u) D_h u + 2\sigma L(u_h) u_h, \tag{3.11}$$
$$L(\eta^2(D_h v)_+^2 + \sigma v_h^2) \le 2\eta^2 L(D_h v)(D_h v)_+ + 2\sigma L(v_h) v_h. \tag{3.12}$$

For another version of such estimate for difference quotients, we refer the reader to Lemma 3.5.4. Let us point out that the idea to prove Bernstein estimates for difference quotients has already been presented in an earlier version of [33] (see Section 3 in [32]). However, Lemma 3.1.9 was only proved for the local Laplacian in [32]. We refer to [79] for a version of the Bernstein technique for finite difference operators.

Finally, as we mentioned before, we are able to extend our proof of the aforementioned Bernstein key estimates in several directions. In Subsection 3.7.1, we explain how to modify the Bernstein technique in order to obtain a priori regularity estimates for nonlocal parabolic equations. Moreover, in Subsection 3.7.2, we prove Bernstein key estimates for nonlocal operators with nonsymmetric kernels under the presence of additional drift terms, and in Subsection 3.7.3 we consider nonlocal operators that are not necessarily comparable to the fractional Laplacian.

3.1.4 Outline

This article is structured as follows: In Section 3.2 we present and prove several auxiliary results which will be crucial in the derivation of our main results. Section 3.3 is devoted to the study of Bernstein key estimates for nonlocal operators and contains the proof of our main results, Theorem 3.1.6 and Theorem 3.1.8. Moreover, in Section 3.4, we prove semiconvexity estimates for solutions to nonlocal fully nonlinear equations (see Theorem 3.1.1). The Bernstein key estimates for difference quotients, and in particular Lemma 3.1.9, are established in Section 3.5. In Section 3.6, we prove our main results on the nonlocal obstacle problem, Theorem 3.1.3, and Theorem 3.1.4. By application of the Bernstein technique, we prove our main auxiliary result, the convexity of blow-ups (see Theorem 3.1.5) and establish semiconvexity estimates for solutions (see Theorem 3.1.2). Finally, in Section 3.7, we discuss several extensions of our technique to parabolic equations, nonlocal equations with drifts, and nonlocal operators that are not necessarily comparable to the fractional Laplacian.

3.2 Preliminaries

Let us introduce the classes of operators we will be working with throughout this article.

Definition 3.2.1 (regularity classes)**.** Let L be an integro-differential operator of the form (3.1) where $K : \mathbb{R}^n \to [0, \infty]$ is symmetric, i.e., $K(y) = K(-y)$.

(i) We say that $L \in \mathcal{L}_s(\lambda, \Lambda)$ for some $s \in (0, 1)$ and $0 < \lambda \le \Lambda$ if L satisfies the following ellipticity condition:

$$\lambda|y|^{-n-2s} \le K(y) \le \Lambda|y|^{-n-2s}. \qquad (K_{\asymp})$$

(ii) We say that $L \in \mathcal{L}_s(\lambda, \Lambda; 1)$ if $L \in \mathcal{L}_s(\lambda, \Lambda)$ and satisfies in addition

$$|\nabla K(y)| \le \Lambda|y|^{-1}K(y). \qquad (C^1)$$

(iii) We say that $L \in \mathcal{L}_s(\lambda, \Lambda; 2)$ if $L \in \mathcal{L}_s(\lambda, \Lambda; 1)$ and satisfies in addition

$$|D^2K(y)| \le \Lambda|y|^{-2}K(y). \qquad (C^2)$$

Remark 3.2.2. (i) Notice that if $L \in \mathcal{L}_s(\lambda, \Lambda)$, then $Lu(x_0)$ is well-defined for any $x_0 \in \mathbb{R}^n$ and $u \in C^{2s+\varepsilon}(B_\delta(x_0)) \cap L^\infty(\mathbb{R}^n)$, for some $\varepsilon, \delta > 0$.

(ii) The assumptions (C^1) and (C^2) are common in the study of higher regularity for nonlocal equations and appeared first in [50, 52, 51].

Let us associate L with a bilinear form B, defined by

$$B(u, v)(x) = \int_{\mathbb{R}^n} \Big(u(x) - u(y)\Big)\Big(v(x) - v(y)\Big)K(x - y)\,dy.$$

Sometimes, we write $L = L_K$ or $B = B_K$ in order to emphasize the corresponding kernel through the notation.

We observe the following nonlocal product rule:

Lemma 3.2.3. *Let $s \in (0,1)$ and $L \in \mathcal{L}_s(\lambda, \Lambda)$. Then, for any $u, v \in C^{2s+\varepsilon}(B_\delta(x_0)) \cap L^\infty(\mathbb{R}^n)$ we have*

$$L(uv) = uLv + vLu - B(u,v) \quad \text{in} \quad B_\delta(x_0).$$

In particular,

$$L(u^2) = 2uLu - B(u,u) \quad \text{in} \quad B_\delta(x_0).$$

Proof. We compute

$$u(x)v(x) - u(y)v(y) = u(x)(v(x) - v(y)) + v(x)(u(x) - u(y)) - (u(x) - u(y))(v(x) - v(y)),$$

and the result follows. □

3.3 Key estimates for the Bernstein technique

In this section we establish the key estimates for smooth functions. For this, we will need the following result, which allows us to separately consider the singularity at the origin and the behavior at infinity.

Lemma 3.3.1 (Kernel decomposition)**.** *Let $s \in (0,1)$, $L \in \mathcal{L}_s(\lambda, \Lambda; 1)$ and let $\varepsilon \in (0,1)$. Then, there exist $K_1, K_2 : \mathbb{R}^n \to [0, \infty]$ such that $K = K_1 + K_2$ and the following properties hold true:*

(i) $\mathrm{supp}(K_1) \subset B_\varepsilon$,

(ii) $K_1 \equiv K$ *in* $B_{\varepsilon/2}$,

(iii) $K_1 \le K$, $K_2 \le K$,

(iv) $\mathrm{supp}(K_2) \subset \mathbb{R}^n \setminus B_{\varepsilon/2}$,

(v) $|\nabla K_2| \le c_1 \varepsilon^{-1} K$,

(vi) $c_2 \mu_{K_2}(\mathbb{R}^n) \le \mu_K(\mathbb{R}^n \setminus B_{\varepsilon/2}) \le c_3 \mu_{K_2}(\mathbb{R}^n)$,

where we denote $\mu_K = K(y)\,\mathrm{d}y$ and $\mu_{K_2} = K_2(y)\,\mathrm{d}y$, and the constants $c_1, c_2, c_3 > 0$ depend only on n, s, λ, Λ, but not on ε.

Proof. Let $\psi \in C^\infty([0,\infty))$ be a cutoff function satisfying $0 \le \psi \le 1$, $\psi \equiv 0$ in $B_{1/2}$ and $\psi \equiv 1$ in $\mathbb{R}^n \setminus B_1$. Moreover, assume that $|\psi'| \le 4$. We define

$$K_1(y) = \left[1 - \psi\left(\frac{|y|}{\varepsilon}\right)\right] K(y), \quad K_2(y) = \psi\left(\frac{|y|}{\varepsilon}\right) K(y).$$

Then, clearly, $K = K_1 + K_2$ and properties (i), (ii), (iii), and (iv) follow immediately by construction. Moreover, note that

$$|\nabla K_2(y)| \le \varepsilon^{-1} |\psi'(|y|/\varepsilon)| K(y) + \psi(|y|/\varepsilon) |\nabla K(y)| \le 4\varepsilon^{-1} K(y) + 2\Lambda \varepsilon^{-1} K(y),$$

where we used (iv) and (C^1). This proves (v). Note that the first estimate in (vi) is a direct consequence of (iii) and (iv). To show the second inequality in (vi), we compute using $(K_\asymp)$

$$\mu_K(\mathbb{R}^n \setminus B_{\varepsilon/2}) \le c\varepsilon^{-2s} \le c \int_{B_{2\varepsilon} \setminus B_\varepsilon} K(y)\,\mathrm{d}y = \mu_{K_2}(B_{2\varepsilon} \setminus B_\varepsilon) \le c\mu_{K_2}(\mathbb{R}^n),$$

which concludes the proof. □

We also need the following simple observation concerning cutoff functions.

Lemma 3.3.2. *Let $s \in (0,1)$ and let K be symmetric, with*

$$K(y) \leq \Lambda |y|^{-n-2s}, \quad \mathrm{supp}(K) \subset B_\varepsilon$$

for some $\Lambda > 0$ and $\varepsilon \in (0,1)$. Let $\eta \in C^{1,1}(B_1)$. Then, for any $x \in B_1$

$$L(\eta^2)(x) \leq c_1 \|D^2\eta^2\|_{L^\infty(B_\varepsilon(x))}\varepsilon^{2-2s},$$
$$B(\eta,\eta)(x) \leq c_2 \|\nabla\eta\|^2_{L^\infty(B_\varepsilon(x))}\varepsilon^{2-2s},$$

where $c_1, c_2 > 0$ are constants depending only on n, s, Λ.

Proof. For the first estimate we compute

$$\begin{aligned} L(\eta^2)(x) &= \int_{B_\varepsilon(x)} (\eta^2(x) - \eta^2(y) + \nabla\eta^2(x)(x-y))K(x-y)\,\mathrm{d}y \\ &\leq \Lambda \|D^2\eta^2\|_{L^\infty(B_\varepsilon(x))} \int_{B_\varepsilon(x)} |x-y|^{2-n-2s}\,\mathrm{d}y \\ &\leq c\Lambda \|D^2\eta^2\|_{L^\infty(B_\varepsilon(x))}\varepsilon^{2-2s}. \end{aligned}$$

Note that we used the even symmetry of K in the first identity. For the second estimate, we observe

$$\begin{aligned} B(\eta,\eta)(x) &= \int_{B_\varepsilon(x)} |\eta(x)-\eta(y)|^2 K(x-y)\,\mathrm{d}y \\ &\leq \Lambda \|\nabla\eta\|^2_{L^\infty(B_\varepsilon(x))} \int_{B_\varepsilon(x)} |x-y|^{2-n-2s}\,\mathrm{d}y \\ &\leq c\Lambda \|\nabla\eta\|^2_{L^\infty(B_\varepsilon(x))}\varepsilon^{2-2s}. \end{aligned}$$

□

With this at hand, we can now start the proof of our key estimates in Theorems 3.1.6 and 3.1.8.

3.3.1 First order estimates

The goal of this section is to establish the key estimate for the Bernstein technique (see Theorem 3.1.6), which will be used in order to prove first derivative estimates.

Before we prove Theorem 3.1.6 let us list two equivalent formulations of (3.8) that will turn out to be more convenient to prove:

Remark 3.3.3. The following two estimates are equivalent to (3.8):

$$L(\eta^2)(\partial_e u)^2 - B(\eta^2, (\partial_e u)^2) \leq \eta^2 B(\partial_e u, \partial_e u) + \sigma B(u,u), \tag{3.13}$$

$$\int_{\mathbb{R}^n} (\eta^2(x) - \eta^2(y))(\partial_e u(y))^2 K(x-y)\,\mathrm{d}y \leq \eta^2(x) B(\partial_e u, \partial_e u)(x) + \sigma B(u,u)(x). \tag{3.14}$$

This can be seen as follows: With the help of the nonlocal product rule Lemma 3.2.3 we compute:

$$L(\eta^2(\partial_e u)^2 + \sigma u^2) = L(\eta^2)(\partial_e u)^2 + 2\eta^2 L(\partial_e u)\partial_e u - \eta^2 B(\partial_e u, \partial_e u) - B(\eta^2, (\partial_e u)^2) \\ + 2\sigma L(u)u - \sigma B(u,u).$$

Therefore, (3.8) is equivalent to (3.13). Moreover, the left hand side in (3.13) can be rewritten as follows:

$$L(\eta^2)(\partial_e u)^2 - B(\eta^2, (\partial_e u)^2) = \int_{\mathbb{R}^n} (\eta^2(x) - \eta^2(y))(\partial_e u(y))^2 K(x-y)\,\mathrm{d}y.$$

This is a simple consequence of the following identity:

$$(\eta^2(x) - \eta^2(y))(\partial_e u(x))^2 - (\eta^2(x) - \eta^2(y))((\partial_e u(x))^2 - (\partial_e u(y))^2) = (\eta^2(x) - \eta^2(y))(\partial_e u(y))^2.$$

Thus, (3.8) and (3.13) are both equivalent to (3.14).

Moreover, the following interpolation inequality will turn out to be crucial in the proof of Theorem 3.1.6.

Lemma 3.3.4. *Let $s \in (0,1)$ and $\delta > 0$. Assume that $K : \mathbb{R}^n \to [0,\infty]$ satisfies for some $0 < \lambda \le \Lambda$:*

$$\lambda |y|^{-n-2s} \le K(y) \le \Lambda |y|^{-n-2s} \quad \forall y \in B_\delta, \tag{3.15}$$

$$|\nabla K(y)| \le \Lambda |y|^{-1} K(y) \quad \forall y \in B_\delta. \tag{3.16}$$

Then, for every $x \in \mathbb{R}^n$ and $u \in C^{0,1}(B_\delta(x))$ it holds

$$\Big(\partial_e u(x)\Big)^2 \le \delta^{2s} B(\partial_e u, \partial_e u)(x) + c\delta^{2s-2} B(u,u)(x),$$

where $c = c(n,s,\lambda,\Lambda) > 0$ does not depend on δ.

Proof. First, given any $\delta > 0$, we construct an auxiliary kernel $K_\delta : \mathbb{R}^n \to [0,\infty]$ satisfying the following properties:

(1) $K_\delta^2(y) \le c_1 K(y)|y|^2$ for $y \in B_\delta$,

(2) $|\nabla K_\delta(y)|^2 \le c_2 K(y)$ for $y \in B_\delta$,

(3) $\mathrm{supp}(K_\delta) \subset B_\delta(0)$,

(4) $c_3 \delta^{\frac{n}{2}-s+1} \le \mu_{K_\delta}(B_\delta) \le c_4 \delta^{\frac{n}{2}-s+1}$,

where we define $\mu_{K_\delta} = K_\delta(y)\,\mathrm{d}y$, and $c_1, c_2, c_3, c_4 > 0$ are constants depending only on n, s, λ, Λ. To do so, we proceed in the same way as in the proof of Lemma 3.3.1. Indeed, let $\psi \in C^\infty([0,\infty))$ by a cutoff function satisfying $\psi \equiv 1$ in $B_{1/2}$, $\psi \equiv 0$ in $\mathbb{R}^n \setminus B_1$, $0 \le \psi \le 1$ and $|\psi'| \le 4$. Then, we define

$$K_\delta(y) = \psi\left(\frac{|y|}{\delta}\right) K(y)|y|^{\frac{n}{2}+s+1}$$

and observe that the properties (1), (3) and (4) follow immediately from the construction and (3.15). To prove (2), we compute for $y \in B_\delta$, using (3.15) and (3.16)

$$|\nabla K_\delta(y)|^2 \lesssim \left(\frac{|y|}{\delta}\right)^2 \left|\psi'\left(\frac{|y|}{\delta}\right)\right|^2 K^2(y)|y|^{n+2s} + \psi^2\left(\frac{|y|}{\delta}\right)\left(|\nabla K(y)|^2|y|^{n+2s+2} + K^2(y)|y|^{n+2s}\right) \\ \lesssim K(y).$$

Having constructed K_δ, let us turn to proving the desired interpolation estimate. We compute, using (4), and the notation $\mu_{K_\delta}(x, \mathrm{d}y) = K_\delta(x-y)\,\mathrm{d}y$:

$$
\begin{aligned}
\partial_e u(x) &= \fint_{B_\delta(x)} (\partial_e u(x) - \partial_e u(y))\mu_{K_\delta}(x, \mathrm{d}y) + \fint_{B_\delta(x)} \partial_e u(y)\mu_{K_\delta}(x, \mathrm{d}y) \\
&= \frac{|B_\delta(x)|}{\mu_{K_\delta}(B_\delta(x))} \left[\fint_{B_\delta(x)} (\partial_e u(x) - \partial_e u(y))K_\delta(x-y)\,\mathrm{d}y + \fint_{B_\delta(x)} \partial_e u(y) K_\delta(x-y)\,\mathrm{d}y \right] \\
&\lesssim \delta^{\frac{n}{2}+s-1} \fint_{B_\delta(x)} (\partial_e u(x) - \partial_e u(y))K_\delta(x-y)\,\mathrm{d}y + \delta^{\frac{n}{2}+s-1} \fint_{B_\delta(x)} \partial_e u(y) K_\delta(x-y)\,\mathrm{d}y.
\end{aligned}
$$

By Jensen's inequality, integration by parts, and (1), (2), and (3):

$$
\begin{aligned}
|\partial_e u(x)|^2 &\lesssim \delta^{n+2s-2} \left[\fint_{B_\delta(x)} (\partial_e u(x) - \partial_e u(y))^2 K_\delta^2(x-y)\,\mathrm{d}y + \left(\fint_{B_\delta(x)} \partial_e u(y) K_\delta(x-y)\,\mathrm{d}y \right)^2 \right] \\
&\lesssim \delta^{n+2s-2} \left[\fint_{B_\delta(x)} (\partial_e u(x) - \partial_e u(y))^2 K_\delta^2(x-y)\,\mathrm{d}y + \left(\fint_{B_\delta(x)} (u(y) - u(x))\partial_e K_\delta(x-y)\,\mathrm{d}y \right)^2 \right] \\
&\lesssim \delta^{2s-2} \left[\int_{B_\delta(x)} (\partial_e u(x) - \partial_e u(y))^2 K(x-y)|x-y|^2\,\mathrm{d}y + \int_{B_\delta(x)} (u(y)-u(x))^2 \left(\partial_e K_\delta(x-y)\right)^2\,\mathrm{d}y \right] \\
&\lesssim \delta^{2s} \int_{B_\delta(x)} (\partial_e u(x) - \partial_e u(y))^2 K(x-y)\,\mathrm{d}y + \delta^{2s-2} \int_{B_\delta(x)} (u(y)-u(x))^2 K(x-y)\,\mathrm{d}y.
\end{aligned}
$$

Therefore, we obtain

$$
\left(\partial_e u(x)\right)^2 \le c_1 \delta^{2s} B(\partial_e u, \partial_e u)(x) + c_2 \delta^{2s-2} B(u,u)(x).
$$

To conclude, we may repeat the computation with $c_1^{-1/2s}\delta$ instead of δ if $c_1 > 1$. □

Remark 3.3.5. Note that if we replace the constants λ, Λ in (3.15) and (3.16) by $\lambda(1-s), \Lambda(1-s)$, where $s \in (s_0, 1)$, then the constant $c > 0$ in Lemma 3.3.4 would only depend on n, s_0, λ, Λ. To see this, one redefines $K_\delta(y) = \sqrt{1-s}\,\psi(|y|/\delta)K(y)|y|^{\frac{n}{2}+s+1}$.

Now, we are in the position to prove the key estimate Theorem 3.1.6.

Proof of Theorem 3.1.6. Throughout the proof, we will denote by $c > 0$ any constant that only depends on n, s, λ, Λ and whose value might change from line to line. Note that we can assume without loss of generality that $\eta(x) > 0$, since otherwise the desired estimate (3.14) is trivially satisfied.

For every x, let us decompose the kernel K into two parts $K = K_1 + K_2$, taking care separately of the behavior at zero and at infinity. In order to do so, we choose a parameter

$$
\varepsilon := \gamma\eta(x) > 0,
$$

where $\gamma = \gamma(\|\eta\|_{C^{1,1}(\mathbb{R}^n)}) \in (0,1)$ will be determined later (see (3.21)), and choose K_1 and K_2 as in Lemma 3.3.1 with respect to ε. Consequently K_1 and K_2 satisfy the properties $(i)-(vi)$.

Step 1: Let us first explain how to treat the integrals involving K_2. To be precise, we will show that for some constant $\sigma_2 = \sigma_2(n, s, \lambda, \Lambda) > 0$:

$$\begin{aligned}\int_{\mathbb{R}^n} &(\eta^2(x) - \eta^2(y))(\partial_e u(y))^2 K_2(x-y)\,\mathrm{d}y \\ &\le \eta^2(x) \int_{\mathbb{R}^n} (\partial_e u(x) - \partial_e u(y))^2 K_2(x-y)\,\mathrm{d}y + \sigma_2 \frac{\eta^2(x)}{\varepsilon^2} B_K(u,u)(x).\end{aligned} \tag{3.17}$$

Note that the claim (3.17) follows if we manage to prove:

$$\begin{aligned}\eta^2(x) &\int_{\mathbb{R}^n} (\partial_e u(y))^2 K_2(x-y)\,\mathrm{d}y \\ &\le \eta^2(x) \int_{\mathbb{R}^n} (\partial_e u(x) - \partial_e u(y))^2 K_2(x-y)\,\mathrm{d}y + \sigma_2 \frac{\eta^2(x)}{\varepsilon^2} B_K(u,u)(x),\end{aligned}$$

which is equivalent to

$$\begin{aligned}2\partial_e u(x)\eta^2(x) &\int_{\mathbb{R}^n} \partial_e u(y) K_2(x-y)\,\mathrm{d}y \\ &\le (\partial_e u(x))^2 \eta^2(x) \int_{\mathbb{R}^n} K_2(x-y)\,\mathrm{d}y + \sigma_2 \frac{\eta^2(x)}{\varepsilon^2} B_K(u,u)(x).\end{aligned} \tag{3.18}$$

Note that the first term on the right hand side in (3.18) is finite, since K_2 is integrable due to (iii) and (iv) in Lemma 3.3.1. To prove (3.18), we introduce the measure $\mu_{K_2}(x, \mathrm{d}y) = K_2(x-y)\,\mathrm{d}y$ and use the Young's inequality as follows:

$$\begin{aligned}2\partial_e u(x)\eta^2(x) \int_{\mathbb{R}^n} \partial_e u(y) K_2(x-y)\,\mathrm{d}y &\le \mu_{K_2}(x, \mathbb{R}^n)(\partial_e u(x))^2 \eta^2(x) \\ &\quad + \eta^2(x) \mu_{K_2}(x, \mathbb{R}^n)^{-1} \left(\int_{\mathbb{R}^n} \partial_e u(y) \mu_{K_2}(x, \mathrm{d}y) \right)^2 \\ &=: J_1 + J_2.\end{aligned}$$

For J_1 we obtain

$$J_1 = \mu_{K_2}(x, \mathbb{R}^n)(\partial_e u(x))^2 \eta^2(x) = (\partial_e u(x))^2 \eta^2(x) \int_{\mathbb{R}^n} K_2(x-y)\,\mathrm{d}y,$$

which coincides with the first term on the right hand side of (3.18). In order to estimate J_2, let us recall that by (iv), (v), and (vi) in Lemma 3.3.1, we have

$$\operatorname{supp}(K_2(x - \cdot)) \subset \mathbb{R}^n \setminus B_{\varepsilon/2}(x), \quad |\nabla K_2(x - \cdot)| \le c\varepsilon^{-1} K(x - \cdot), \quad \frac{\mu_K(x, \mathbb{R}^n \setminus B_{\varepsilon/2}(x))}{\mu_{K_2}(x, \mathbb{R}^n)} \le c.$$

Thus, using integration by parts and Jensen's inequality:

$$\begin{aligned}
J_2 &= \eta^2(x)\mu_{K_2}(x,\mathbb{R}^n)^{-1}\left(\int_{\mathbb{R}^n}(u(y)-u(x))\partial_e K_2(x-y)\,\mathrm{d}y\right)^2\\
&\le c\frac{\eta^2(x)}{\varepsilon^2}\mu_{K_2}(x,\mathbb{R}^n)^{-1}\left(\int_{\mathbb{R}^n\setminus B_{\varepsilon/2}(x)}|u(y)-u(x)|K(x-y)\,\mathrm{d}y\right)^2\\
&= c\frac{\eta^2(x)}{\varepsilon^2}\frac{\mu_K(x,\mathbb{R}^n\setminus B_{\varepsilon/2}(x))^2}{\mu_{K_2}(x,\mathbb{R}^n)}\left(\fint_{\mathbb{R}^n\setminus B_{\varepsilon/2}(x)}|u(y)-u(x)|\mu_K(x,\,\mathrm{d}y)\right)^2\\
&\le c\frac{\eta^2(x)}{\varepsilon^2}\int_{\mathbb{R}^n}(u(x)-u(y))^2K(x-y)\,\mathrm{d}y.
\end{aligned}$$

Altogether, by combining the estimates for J_1 and J_2, we have shown (3.18), and therefore (3.17), as desired.

Step 2: We claim that

$$L_{K_1}(\eta^2)(x)(\partial_e u(x))^2 - B_{K_1}(\eta^2,(\partial_e u)^2)(x) \le \eta^2(x)B_{K_1}(\partial_e u,\partial_e u)(x) + \sigma_1 B_K(u,u)(x), \quad (3.19)$$

where $\sigma_1 = \sigma_1(n,s,\lambda,\Lambda,\|\eta\|_{C^{1,1}(\mathbb{R}^n)}) > 0$ is a constant. Note that by combining (3.19) with (3.17) and using that $\varepsilon = \gamma\eta(x)$, we obtain that for every $x\in\mathbb{R}^n$:

$$L_K(\eta^2)(x)(\partial_e u(x))^2 - B_K(\eta^2,(\partial_e u)^2)(x) \le \eta^2(x)B_K(\partial_e u,\partial_e u)(x) + \sigma B_K(u,u)(x), \quad (3.20)$$

where $\sigma = \sigma_1 + \sigma_2\gamma^{-2} > 0$. This proves the desired result.

Let us now prove (3.19). Recall that $\mathrm{supp}(K_1(x-\cdot)) \subset B_\varepsilon(x)$ by (i) in Lemma 3.3.1. By Young's again, we compute for $A = \frac{1}{8}(\|\nabla\eta\|_{L^\infty(\mathbb{R}^n)}+2)^{-2} > 0$:

$$\begin{aligned}
-B_{K_1}(\eta^2,(\partial_e u)^2)(x) &= \int_{B_\varepsilon(x)}(\eta^2(x)-\eta^2(y))[(\partial_e u(x))^2-(\partial_e u(y))^2]K_1(x-y)\,\mathrm{d}y\\
&\le A\int_{B_\varepsilon(x)}(\eta(x)+\eta(y))^2(\partial_e u(x)-\partial_e u(y))^2K_1(x-y)\,\mathrm{d}y\\
&\quad+\frac{1}{4A}\int_{B_\varepsilon(x)}(\eta(x)-\eta(y))^2(\partial_e u(x)+\partial_e u(y))^2K_1(x-y)\,\mathrm{d}y\\
&=: I_1 + I_2.
\end{aligned}$$

For I_1, we use that for $y\in B_\varepsilon(x)$

$$|\eta(x)-\eta(y)| \le \|\nabla\eta\|_{L^\infty(B_\varepsilon(x))}|x-y| \le \|\nabla\eta\|_{L^\infty(\mathbb{R}^n)}\varepsilon = \gamma\|\nabla\eta\|_{L^\infty(\mathbb{R}^n)}\eta(x),$$

and therefore

$$(\eta(x)+\eta(y))^2 \le (2+\gamma\|\nabla\eta\|_{L^\infty(B_\varepsilon(x))})^2\eta(x)^2 \le (2+\|\nabla\eta\|_{L^\infty(B_\varepsilon(x))})^2\eta(x)^2.$$

We obtain

$$I_1 \le A(\|\nabla\eta\|_{L^\infty(\mathbb{R}^n)}+2)^2\eta^2(x)\int_{B_\varepsilon(x)}(\partial_e u(x)-\partial_e u(y))^2K_1(x-y)\,\mathrm{d}y \le \frac{\eta^2(x)}{8}B_{K_1}(\partial_e u,\partial_e u)(x).$$

For I_2, we make use of the following algebraic inequality

$$(a+b)^2 \le (a+b)^2 + (3a-b)^2 = 8a^2 + 2(a-b)^2$$

and apply it to $a = \partial_e u(x)$ and $b = \partial_e u(y)$. This yields, for some $c_1 > 0$,

$$\begin{aligned} I_2 &\le 2A^{-1}(\partial_e u(x))^2 \int_{B_\varepsilon(x)} (\eta(x)-\eta(y))^2 K_1(x-y)\,\mathrm{d}y \\ &\quad + \frac{1}{2A}\int_{B_\varepsilon(x)} (\eta(x)-\eta(y))^2(\partial_e u(x)-\partial_e u(y))^2 K_1(x-y)\,\mathrm{d}y \\ &\le \left[2c_1 A^{-1}\gamma^{2-2s}\|\nabla\eta\|^2_{L^\infty(\mathbb{R}^n)}\right]\eta^{2-2s}(x)(\partial_e u(x))^2 \\ &\quad + \left[(2A)^{-1}\|\nabla\eta\|^2_{L^\infty(\mathbb{R}^n)}\gamma^2\right]\eta^2(x) B_{K_1}(\partial_e u, \partial_e u)(x) \\ &= I_{2,1} + I_{2,2}, \end{aligned}$$

where we applied Lemma 3.3.2. By choosing

$$\gamma = (4A^{-1}\|\nabla\eta\|^2_{L^\infty(\mathbb{R}^n)})^{-\frac{1}{2}} \wedge 1, \tag{3.21}$$

we estimate

$$I_{2,2} \le \frac{\eta^2(x)}{8} B_{K_1}(\partial_e u, \partial_e u)(x).$$

We apply Lemma 3.3.4 to K_1 with $\delta = [8c_1 A^{-1}\gamma^{2-2s}\|\nabla\eta\|^2_{L^\infty(\mathbb{R}^n)}]^{-\frac{1}{2s}}\eta(x) \wedge \frac{\gamma}{2}\eta(x)$ in order to estimate $I_{2,1}$. Note that K_1 satisfies the assumptions of Lemma 3.3.4 due to Lemma 3.3.1. This yields

$$I_{2,1} \le \frac{\eta^2(x)}{4} B_{K_1}(\partial_e u, \partial_e u)(x) + \sigma_{1,1} B_{K_1}(u,u)(x),$$

where, for $c_2 > 0$ being the constant in Lemma 3.3.4,

$$\sigma_{1,1} = \frac{c_2}{4}\left[8c_1 A^{-1}\gamma^{2-2s}\|\nabla\eta\|^2_{L^\infty(\mathbb{R}^n)}\right]^{\frac{1}{s}} \vee 2^{3-2s} c_1 c_2 A^{-1}\|\nabla\eta\|^2_{L^\infty(\mathbb{R}^n)} > 0.$$

By combination of the estimates for I_1, $I_{2,1}$ and $I_{2,2}$, we obtain

$$-B_{K_1}(\eta^2, (\partial_e u)^2)(x) \le \frac{1}{2}\eta^2(x) B_{K_1}(\partial_e u, \partial_e u)(x) + \sigma_{1,1} B(u,u)(x). \tag{3.22}$$

Finally, we observe that for some $c_3 > 0$, by Lemma 3.3.2

$$L_{K_1}(\eta^2)(x) \le c_3\|D^2\eta^2\|_{L^\infty(\mathbb{R}^n)}\varepsilon^{2-2s} = \left[c_3\|D^2\eta^2\|_{L^\infty(\mathbb{R}^n)}\gamma^{2-2s}\right]\eta^{2-2s}(x),$$

and apply Lemma 3.3.4 to K_1 with $\delta = [4c_3\|D^2\eta^2\|_{L^\infty(\mathbb{R}^n)}\gamma^{2-2s}]^{-\frac{1}{2s}}\eta(x) \wedge \frac{\gamma}{2}\eta(x)$ and obtain

$$L_{K_1}(\eta^2)(x)(\partial_e u(x))^2 \le \frac{1}{4}\eta^2(x) B_{K_1}(\partial_e u, \partial_e u)(x) + \sigma_{1,2} B(u,u)(x),$$

where, for $c_2 > 0$ being the constant in Lemma 3.3.4

$$\sigma_{1,2} = \frac{c_2}{4}\left[4c_3\|D^2\eta^2\|_{L^\infty(\mathbb{R}^n)}\gamma^{2-2s}\right]^{\frac{1}{s}} \vee 2^{2-2s} c_2 c_3\|D^2\eta^2\|_{L^\infty(\mathbb{R}^n)} > 0.$$

Consequently, we obtain (3.19) with $\sigma_1 = \sigma_{1,1} + \sigma_{1,2}$, as desired. □

Remark 3.3.6. Note that if we choose $L \in \mathcal{L}_s((1-s)\lambda, (1-s)\Lambda; 1)$ and $s \ge s_0$ for some $s_0 \in (0,1)$, then the constant $\sigma = \sigma_1 + \sigma_2\gamma^{-2} > 0$ in (3.8) will depend only on n, s_0, λ, Λ. This is because the constants in Step 1 will not depend on s, and the constant in Step 2 will be of the form

$$\sigma_1 = \sigma_{1,1} + \sigma_{1,2},$$

with

$$\sigma_{1,1} = \frac{c_2}{4}\left[8c_1 A^{-1}\gamma^{2-2s}\|\nabla\eta\|^2_{L^\infty(\mathbb{R}^n)}\right]^{\frac{1}{s}} \vee 2^{3-2s}c_1c_2A^{-1}\|\nabla\eta\|^2_{L^\infty(\mathbb{R}^n)},$$
$$\sigma_{1,2} = \frac{c_2}{4}\left[4c_3\|D^2\eta^2\|_{L^\infty(\mathbb{R}^n)}\gamma^{2-2s}\right]^{\frac{1}{s}} \vee 2^{2-2s}c_2c_3\|D^2\eta^2\|_{L^\infty(\mathbb{R}^n)},$$

and these expressions remain bounded as $s \nearrow 1$. The constants $\gamma, \sigma_2, c_1, c_2, c_3, A > 0$ depend only on n, s_0, λ, Λ.

3.3.2 One-sided second order estimates

In this section, we prove another key estimate, reminiscent of Theorem 3.1.6, which will allow us to prove one-sided second derivative bounds for solutions to certain PDEs driven by L.

Remark 3.3.7. The following two estimates are equivalent to (3.10):

$$\begin{aligned} L(\eta^2)(\partial_e v)_+^2 - B(\eta^2, (\partial_e v)_+^2) &\le \eta^2 B((\partial_e v)_+, (\partial_e v)_+) + 2\eta^2(\partial_e v)_+\left[L(\partial_e v) - L((\partial_e v)_+)\right] + \sigma B(v,v) \\ &= \eta^2 B((\partial_e v)_+, (\partial_e v)_+) - 2\eta^2(\partial_e v)_+ L((\partial_e v)_-) + \sigma B(v,v), \end{aligned} \tag{3.23}$$

$$\begin{aligned} &\int_{\mathbb{R}^n}(\eta^2(x) - \eta^2(y))((\partial_e v(y))_+)^2 K(x-y)\,\mathrm{d}y \\ &\qquad \le \eta^2 B((\partial_e v)_+, (\partial_e v)_+)(x) - 2\eta^2(\partial_e v)_+ L((\partial_e v)_-)(x) + \sigma B(v,v)(x). \end{aligned} \tag{3.24}$$

The proof goes by the same arguments as for (3.13) and (3.14).

The strategy of our proof is similar to the one of the first key estimate Theorem 3.1.6. We will again split the kernel into two parts K_1 and K_2, taking care of the singularity at zero, and the decay at infinity, respectively. In order to treat K_1 we need to prove an interpolation inequality similar to Lemma 3.3.4:

Lemma 3.3.8. *Let $\delta > 0$. Assume that K satisfies (3.15) and (3.16). Then, for every $x \in \mathbb{R}^n$ and $v \in C^{0,1}(B_\delta(x))$ it holds*

$$(\partial_e v(x))_+^2 \le \delta^{2s}B((\partial_e v)_+, (\partial_e v)_+)(x) - \delta^{2s}L((\partial_e v)_-)(x)(\partial_e v)_+(x) + c\delta^{2s-2}B(v,v)(x).$$

where $c = c(n, s, \lambda) > 0$ does not depend on δ.

Proof. Assume that $\partial_e v(x) > 0$, otherwise the estimate is trivial. Let K_δ be as in the proof of Lemma 3.3.4. By following the proof of Lemma 3.3.4, we obtain

$$\begin{aligned} \partial_e v(x) &\le c\delta^{\frac{n}{2}+s-1}\left[\fint_{B_\delta(x)}(\partial_e v(x) - \partial_e v(y))K_\delta(x-y)\,\mathrm{d}y + \fint_{B_\delta(x)}\partial_e v(y)K_\delta(x-y)\,\mathrm{d}y\right] \\ &= c\delta^{\frac{n}{2}+s-1}\left[\fint_{B_\delta(x)}((\partial_e v(x))_+ - (\partial_e v(y))_+)K_\delta(x-y)\,\mathrm{d}y + \fint_{B_\delta(x)}(\partial_e v(y))_- K_\delta(x-y)\,\mathrm{d}y\right] \\ &\quad + c\delta^{\frac{n}{2}+s-1}\fint_{B_\delta(x)}\partial_e v(y)K_\delta(x-y)\,\mathrm{d}y. \end{aligned}$$

By multiplication of the aforementioned estimate with $\partial_e v(x)$ on both sides, and applying Young's inequality, we obtain

$$\begin{aligned}(\partial_e v(x))^2 \le c\left(\delta^{\frac{n}{2}+s-1} \fint_{B_\delta(x)} ((\partial_e v(x))_+ - (\partial_e v(y))_+) K_\delta(x-y)\,\mathrm{d}y\right)^2 \\ + c\delta^{\frac{n}{2}+s-1}\partial_e v(x) \fint_{B_\delta(x)} (\partial_e v(y))_- K_\delta(x-y)\,\mathrm{d}y \\ + c\left(\delta^{\frac{n}{2}+s-1} \fint_{B_\delta(x)} \partial_e v(y) K_\delta(x-y)\,\mathrm{d}y\right)^2 + \frac{1}{2}(\partial_e v(x))^2.\end{aligned}$$

From here, with the first and third term, we proceed as in the proof of Lemma 3.3.4. The fourth term can be absorbed to the left hand side. This yields

$$\begin{aligned}(\partial_e v(x))^2 \lesssim \delta^{2s} \int_{B_\delta(x)} ((\partial_e v(x))_+ - (\partial_e v(y))_+)^2 K(x-y)\,\mathrm{d}y \\ + \delta^{\frac{n}{2}+s-1}\partial_e v(x) \fint_{B_\delta(x)} (\partial_e v(y))_- K_\delta(x-y)\,\mathrm{d}y \\ + \delta^{2s-2} \int_{B_\delta(x)} (v(y)-v(x))^2 K(x-y)\,\mathrm{d}y.\end{aligned}$$

In order to treat the second term on the right hand side, we recall the definition of K_δ and deduce the following estimate:

$$\begin{aligned}&\delta^{\frac{n}{2}+s-1}\partial_e v(x) \fint_{B_\delta(x)} (\partial_e v(y))_- K_\delta(x-y)\,\mathrm{d}y \\ &\qquad \le \delta^{-\frac{n}{2}+s-1}\partial_e v(x) \int_{B_\delta(x)} (\partial_e v(y))_- K(x-y)|x-y|^{\frac{n}{2}+s+1}\,\mathrm{d}y \\ &\qquad \le \delta^{2s}\partial_e v(x) \int_{B_\delta(x)} (\partial_e v(y))_- K(x-y)\,\mathrm{d}y \\ &\qquad = -\delta^{2s}\partial_e v(x) \int_{B_\delta(x)} ((\partial_e v(x))_- - (\partial_e v(y))_-) K(x-y)\,\mathrm{d}y \\ &\qquad \le -\delta^{2s} L((\partial_e v)_-)(x)\partial_e v(x).\end{aligned} \tag{3.25}$$

Altogether, taking δ smaller if necessary as in Lemma 3.3.4 we obtain the desired result. □

We are now ready to give the proof of Theorem 3.1.8.

Proof of Theorem 3.1.8. Let $\varepsilon = \gamma\eta(x)$ as in the proof of Theorem 3.1.6. Moreover, define K_1 and K_2 as in Lemma 3.3.1 with respect to ε.

Step 1: Let us first explain how to treat the integrals involving K_2. We show that for some uniform constant $\sigma_2 > 0$:

$$\begin{aligned}\int_{\mathbb{R}^n} (\eta^2(x) - \eta^2(y))(\partial_e v(y))_+^2 K_2(x-y)\,\mathrm{d}y \le \eta^2(x) B_{K_2}((\partial_e v)_+, (\partial_e v)_+)(x) \\ - 2\eta^2(x)(\partial_e v)_+(x) L_{K_2}((\partial_e v)_-)(x) + \sigma_2 \frac{\eta^2(x)}{\varepsilon^2} B_K(v,v)(x).\end{aligned} \tag{3.26}$$

The proof follows by the same idea as in Theorem 3.1.6. Note that because our key estimates contain positive parts, there appears the additional term $-2\eta^2(x)(\partial_e v)_+(x)L_{K_2}((\partial_e v)_-)(x)$ in (3.26). Since this term is nonnegative, we can compensate the possible smallness of the other terms on the right hand side due to the consideration of positive parts.

The estimate (3.26) follows if we manage to prove:

$$\eta^2(x)\int_{\mathbb{R}^n}(\partial_e v(y))_+^2 K_2(x-y)\,\mathrm{d}y \le \eta^2(x)\int_{\mathbb{R}^n}((\partial_e v(x))_+ - (\partial_e v(y))_+)^2 K_2(x-y)\,\mathrm{d}y \\ - 2\eta^2(x)(\partial_e v(x))_+ L_{K_2}((\partial_e v)_-)(x) + c\frac{\eta^2(x)}{\varepsilon^2}B_K(v,v)(x),$$

which is equivalent to

$$2(\partial_e v(x))_+\eta^2(x)\int_{\mathbb{R}^n}(\partial_e v(y))_+ K_2(x-y)\,\mathrm{d}y \le (\partial_e v(x))_+^2\eta^2(x)\int_{\mathbb{R}^n}K_2(x-y)\,\mathrm{d}y \\ + 2(\partial_e v(x))_+\eta^2(x)\int_{\mathbb{R}^n}(\partial_e v(y))_- K_2(x-y)\,\mathrm{d}y + c\frac{\eta^2(x)}{\varepsilon^2}B_K(v,v)(x). \tag{3.27}$$

Here, we used that $(\partial_e v(x))_+(\partial_e v(x))_- = 0$, and hence

$$-2\eta^2(x)(\partial_e v(x))_+ L_{K_2}((\partial_e v)_-)(x) = 2\eta^2(x)(\partial_e v(x))_+\int_{\mathbb{R}^n}((\partial_e v(y))_- - (\partial_e v(x))_-)K_2(x-y)\,\mathrm{d}y \\ = 2\eta^2(x)(\partial_e v(x))_+\int_{\mathbb{R}^n}(\partial_e v(y))_- K_2(x-y)\,\mathrm{d}y.$$

To prove (3.27), we introduce the measure $\mu_{K_2}(x,\,\mathrm{d}y) = K_2(x-y)\,\mathrm{d}y$ and estimate

$$\begin{aligned} &2(\partial_e v(x))_+\eta^2(x)\int_{\mathbb{R}^n}(\partial_e v(y))_+ K_2(x-y)\,\mathrm{d}y \\ &= 2(\partial_e v(x))_+\eta^2(x)\int_{\mathbb{R}^n}(\partial_e v(y))_- K_2(x-y)\,\mathrm{d}y + 2(\partial_e v(x))_+\eta^2(x)\int_{\mathbb{R}^n}\partial_e v(y)K_2(x-y)\,\mathrm{d}y \\ &\le 2(\partial_e v(x))_+\eta^2(x)\int_{\mathbb{R}^n}(\partial_e v(y))_- K_2(x-y)\,\mathrm{d}y + \mu_{K_2}(x,\mathbb{R}^n)((\partial_e v(x))_+)^2\eta^2(x) \\ &\quad + \eta^2(x)\mu_{K_2}(x,\mathbb{R}^n)^{-1}\left(\int_{\mathbb{R}^n}\partial_e v(y)\mu_{K_2}(x,\,\mathrm{d}y)\right)^2 \\ &=: J_0 + J_1 + J_2. \end{aligned}$$

Note that J_0 already coincides with second term on the right hand side of (3.27). In order to estimate J_1 and J_2 we proceed precisely by the same arguments as in the estimation of J_1 and J_2 in Step 1 of the proof of Theorem 3.1.6. This proves (3.27), and therefore (3.26), as desired.

Step 2: We claim that

$$L_{K_1}(\eta^2)(x)(\partial_e v(x))_+^2 - B_{K_1}(\eta^2,(\partial_e v)_+^2)(x) \le \eta^2(x)B_{K_1}((\partial_e v)_+,(\partial_e v)_+)(x) \\ - 2\eta^2(x)(\partial_e v(x))_+ L_{K_1}((\partial_e v)_-)(x) + \sigma_1 B_K(v,v)(x), \tag{3.28}$$

where $\sigma_1 > 0$ is a constant.
To establish (3.28), the same proof as for (3.19) in Step 2 for Theorem 3.1.6 goes through. One

only needs to replace $\partial_e u$ by $(\partial_e v)_+$ to deduce (3.22). Hence, we only need to prove

$$\begin{aligned} L_{K_1}(\eta^2)(x)(\partial_e v(x))_+^2 \le \frac{1}{2}\eta^2(x)B_{K_1}((\partial_e v)_+,(\partial_e v)_+)(x) \\ - 2\eta^2(x)(\partial_e v(x))_+ L_K((\partial_e v)_-)(x) + \sigma_{1,2}B(v,v)(x). \end{aligned}$$

If $\partial_e v(x) \le 0$ the estimate is trivial. Otherwise, following the computations in Step 2 using Lemma 3.3.8 instead of Lemma 3.3.4 gives

$$\begin{aligned} L_{K_1}(\eta^2)(x)(\partial_e v(x))_+^2 \le \frac{1}{4}\eta^2(x)B_{K_1}((\partial_e v)_+,(\partial_e v)_+)(x) \\ - \frac{1}{4}\eta^2(x)(\partial_e v(x))_+ L_K((\partial_e v)_-)(x) + \sigma_{1,2}B(v,v)(x), \end{aligned}$$

so, it suffices to note that $L_K((\partial_e v)_-)(x) < 0$ whenever $\partial_e v(x) > 0$.

Note that by combining (3.28) with (3.26) and using that $\varepsilon = \gamma\eta(x)$, we obtain that for every $x \in \mathbb{R}^n$:

$$\begin{aligned} L_K(\eta^2)(x)(\partial_e v(x))^2 - B_K(\eta^2,(\partial_e v)^2)(x) \le \eta^2(x)B_K(\partial_e v,\partial_e v)(x) \\ - 2\eta^2(x)(\partial_e v(x))_+ L_K((\partial_e v)_-)(x) + \sigma B_K(v,v)(x), \end{aligned} \tag{3.29}$$

where $\sigma = \sigma(n, s, \Lambda/\lambda, \|\eta\|_{C^{1,1}(\mathbb{R}^n)}) > 0$. This proves the desired result. □

Let us now prove a key estimate involving second derivatives. It is a straightforward corollary of Theorem 3.1.8.

Corollary 3.3.9. *Let $s \in (0,1)$, $L \in \mathcal{L}_s(\lambda,\Lambda;1)$. Let $\eta \in C_c^{1,1}(B_1)$ be such that $\eta \equiv 1$ in $B_{1/2}$ and $0 \le \eta \le 1$. Moreover, let $\bar\eta \in C_c^\infty(B_{1/4})$ be such that $\bar\eta \equiv 1$ in $B_{1/8}$ and $0 \le \bar\eta \le 1$. Then, there exists $\sigma_0 = \sigma_0(n,s,\lambda,\Lambda,\|\bar\eta\|_{C^{1,1}(\mathbb{R}^n)}) > 0$ such that for every $\sigma \ge \sigma_0$ and $u \in C^{2+2s+\varepsilon}(B_1)$ the following estimate holds true for $\tilde u := \eta u$:*

$$L(\bar\eta^2(-\partial^2_{ee}\tilde u)_+^2 + \sigma(\partial_e \tilde u)^2) \le 2\bar\eta^2 L(-\partial^2_{ee}\tilde u)(-\partial^2_{ee}\tilde u)_+ + 2\sigma L(\partial_e\tilde u)\partial_e\tilde u \quad \text{in} \quad \mathbb{R}^n. \tag{3.30}$$

Proof. The result follows by application of Theorem 3.1.8 with $-\partial_e\tilde u$. □

3.4 Application to fully nonlinear equations

The goal of this section is to establish Theorem 3.1.1. The main tool in our proof is the Bernstein technique for integro-differential operators, which we develop in this article.

First, let us state a more general version of Theorem 3.1.1. To do so, let us introduce the class of fully nonlinear operators $\mathcal{J}_s(\lambda,\Lambda)$:

Definition 3.4.1. We define $\mathcal{J}_s(\lambda,\Lambda)$ to be the set of all operators $\mathcal{I}$ of the form

$$\mathcal{I}u = \inf_{\gamma\in\Gamma}\{L_\gamma u - c_\gamma\},$$

where Γ is an index set and for any $\gamma \in \Gamma$ it holds $L_\gamma \in \mathcal{L}_s(\lambda,\Lambda;2)$, and $c_\gamma \in C^{1,1}$ satisfying

$$\sup_{\gamma\in\Gamma}\|c_\gamma\|_{C^{1,1}} \le \Lambda.$$

We prove the following.

Theorem 3.4.2. *Let $s \in (0,1)$ and $\mathcal{I} \in \mathfrak{I}_s(\lambda,\Lambda)$. Let $u \in C(B_1) \cap L^\infty(\mathbb{R}^n)$ be a viscosity solution to*

$$\mathcal{I}u = 0 \quad in \quad B_1. \tag{3.31}$$

Then,

$$\inf_{B_{1/8}} \partial^2_{ee} u \geq -C \left(\sup_{\gamma\in\Gamma} \|c_\gamma\|_{C^{1,1}(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)} \right), \tag{3.32}$$

where C depends only on n, s, λ, Λ.

Proof. We start by proving that for any smooth solution $u \in C^\infty(B_1) \cap L^\infty(\mathbb{R}^n)$ of an equation of the type (3.31) we have

$$\sup_{B_{1/8}} -\partial^2_{ee} u \leq C \left(\sup_{\gamma\in\Gamma} \|\nabla c_\gamma\|_{L^\infty(B_1)} + \sup_{\gamma\in\Gamma} \|D^2 c_\gamma\|_{L^\infty(B_1)} + \|\partial_e u\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)} \right), \tag{3.33}$$

where $C = C(n,s,\lambda,\Lambda)$.

For this, a key observation is that, by the regularity of u, for every $x \in B_1$ there exist $L_{\gamma(x)} \in \mathcal{L}_s(\lambda,\Lambda;2)$ and $c_{\gamma(x)} \in C^{1,1}$ with $\|c_{\gamma(x)}\|_{C^{1,1}} \leq \Lambda$ such that

$$L_{\gamma(x)} u(x) = c_{\gamma(x)}(x).$$

To see this, we take minimizing sequences (L_n) and (c_n) such that $-L_n u(x) + c_n(x) \searrow 0$. Note that by Arzelà-Ascoli, the sequence (c_n) converges up to a subsequence to $c_{\gamma(x)}$ with the aforementioned properties and moreover, the sequence (L_n) weakly convergences up to a subsequence to $L_{\gamma(x)}$ satisfying $c_{\gamma(x)}(x) = \lim_{n\to\infty} c_n(x) = \lim_{n\to\infty} L_n u(x) = L_{\gamma(x)} u(x)$.

Step 1: We claim that the following holds true in the classical sense:

$$L_{\gamma(\cdot)}(\partial_e u) = \partial_e c_{\gamma(\cdot)}, \quad L_{\gamma(\cdot)}(-\partial^2_{ee} u) \leq -\partial^2_{ee} c_{\gamma(\cdot)} \quad \text{in } B_{3/4}. \tag{3.34}$$

In fact, since we have $L_{\gamma(\cdot)} u \geq c_{\gamma(\cdot)}$ in B_1, we find that for every small enough $h = te$, with $t \in \mathbb{R}$, and any $x \in B_{3/4}$

$$\begin{aligned} L_{\gamma(x)}(D_h u)(x) &= \frac{L_{\gamma(x)} u(x+h) - L_{\gamma(x)} u(x)}{|h|} \geq D_h c_{\gamma(x)}(x), \\ D_h c_{\gamma(x)}(x-h) &\geq \frac{L_{\gamma(x)} u(x-h) - L_{\gamma(x)} u(x)}{-|h|} = L_{\gamma(x)}(-D_{-h} u)(x), \end{aligned}$$

and

$$L_{\gamma(x)}(D_h D_{-h} u)(x) = \frac{2L_{\gamma(x)} u(x) - L_{\gamma(x)} u(x+h) - L_{\gamma(x)} u(x-h)}{|h|^2} \leq D_h D_{-h} c_{\gamma(x)}(x).$$

The aforementioned inequalities hold true in the classical sense since $u \in C^\infty(B_1)$. Therefore, (3.34) follows by taking the limit $|h| \to 0$.

Step 2: Next, we apply Corollary 3.3.9 to u. Let us take two cutoff functions $\eta \in C_c^\infty(B_1)$ and $\bar\eta \in C_c^\infty(B_{1/4})$ with $\eta, \bar\eta \geq 0$ and $\eta \equiv 1$ in $B_{1/2}$, $\bar\eta \equiv 1$ in $B_{1/8}$. We derive that the following estimate holds true in a pointwise sense for any $x \in B_{1/4}$ and $\tilde u = \eta u$:

$$\begin{aligned} &L_{\gamma(x)}(\bar\eta^2(-\partial^2_{ee}u)^2_+ + \sigma(\partial_e \tilde u)^2)(x) \\ &\qquad \leq 2\bar\eta^2 L_{\gamma(x)}(-\partial^2_{ee}\tilde u)(-\partial^2_{ee}\tilde u)_+(x) + 2\sigma L_{\gamma(x)}(\partial_e\tilde u)\partial_e \tilde u(x). \end{aligned} \tag{3.35}$$

Note that

$$L_{\gamma(\cdot)}(\partial_e \tilde u) = L_{\gamma(\cdot)}(\partial_e u) - L_{\gamma(\cdot)}(\partial_e[(1-\eta)u]), \quad L_{\gamma(\cdot)}(\partial^2_{ee}\tilde u) = L_{\gamma(\cdot)}(\partial^2_{ee}u) - L_{\gamma(\cdot)}(\partial^2_{ee}[(1-\eta)u]),$$

and that we can estimate for any $x \in B_{1/4}$ using (C^1)

$$\begin{aligned} |L_{\gamma(x)}(\partial_e[(1-\eta)u])(x)| &= \left| \int_{\mathbb{R}^n \setminus B_{1/4}(x)} \partial_e[(1-\eta)u](y) K_{\gamma(x)}(x-y)\,\mathrm{d}y \right| \\ &= \left| \int_{\mathbb{R}^n \setminus B_{1/4}(x)} (1-\eta)u(y)\partial_e K_{\gamma(x)}(x-y)\,\mathrm{d}y \right| \\ &\leq C\|u\|_{L^\infty(\mathbb{R}^n)}. \end{aligned}$$

An analogous computation, using (C^2) instead of (C^1) yields

$$|L_{\gamma(x)}(\partial^2_{ee}[(1-\eta)u])(x)| \leq C\|u\|_{L^\infty(\mathbb{R}^n)}.$$

By combination of (3.35) with the previous observations and (3.34), we derive

$$\begin{aligned} &L_{\gamma(x)}(\bar\eta^2(-\partial^2_{ee}\tilde u)^2_+ + \sigma(\partial_e\tilde u)^2)(x) \\ &\quad \leq C\bar\eta^2[L_{\gamma(x)}(-\partial^2_{ee}u)(x) + \|u\|_{L^\infty(\mathbb{R}^n)}](-\partial^2_{ee}u)_+(x) + C[L_{\gamma(x)}(\partial_e u) + \|u\|_{L^\infty(\mathbb{R}^n)}]|\partial_e u|(x) \\ &\quad \leq C\bar\eta^2[-\partial^2_{ee}c_\gamma|_{\gamma=\gamma(x)} + \|u\|_{L^\infty(\mathbb{R}^n)}](-\partial^2_{ee}u)_+(x) + C[\partial_e c_\gamma|_{\gamma=\gamma(x)} + \|u\|_{L^\infty(\mathbb{R}^n)}]|\partial_e u|(x) \\ &\quad =: F(x) \end{aligned} \tag{3.36}$$

Step 3: Let us set

$$\phi := \bar\eta^2(-\partial^2_{ee}\tilde u)^2_+ + \sigma(\partial_e \tilde u)^2.$$

By the maximum principle (see for instance [50, 33]),

$$\sup_{B_{1/4}} \phi \leq C \sup_{B_{1/4}} F + \sup_{\mathbb{R}^n \setminus B_{1/4}} \phi.$$

Note that we can estimate by Young's inequality

$$C \sup_{B_{1/4}} F \leq \frac{1}{2} \sup_{B_{1/4}} \phi + \tilde C \sup_{B_{1/4}} \left[\bar\eta^2(\partial^2_{ee}c_{\gamma(\cdot)})^2 + (\partial_e c_{\gamma(\cdot)})^2 + \|u\|^2_{L^\infty(\mathbb{R}^n)} \right].$$

This yields

$$\sup_{B_{1/4}} \phi \leq C \left(\sup_{\gamma \in \Gamma} \|\nabla c_\gamma\|^2_{L^\infty(B_1)} + \sup_{\gamma\in\Gamma} \|D^2 c_\gamma\|^2_{L^\infty(B_1)} + \|u\|^2_{L^\infty(\mathbb{R}^n)} + \sup_{\mathbb{R}^n \setminus B_{1/4}} \phi \right). \tag{3.37}$$

By definition of $\bar{\eta}$ and η, we have

$$\sup_{B_{1/8}}(-\partial^2_{ee}u)^2_+ \leq \sup_{B_{1/8}}\left[(-\partial^2_{ee}u)^2_+ + \sigma(\partial_e u)^2\right] = \sup_{B_{1/8}}\phi \leq \sup_{B_{1/4}}\phi.$$

Moreover,

$$\sup_{\mathbb{R}^n\setminus B_{1/4}}\phi = \sup_{\mathbb{R}^n\setminus B_{1/4}}\left[\sigma(\partial_e\tilde{u})^2\right] \leq \sigma\sup_{B_1}\left[(\partial_e u)^2 + (\partial_e\eta)^2u^2\right].$$

By combination of the previous observations with (3.37), we find

$$\sup_{B_{1/8}}(-\partial^2_{ee}u)^2_+ \leq C\left(\sup_{\gamma\in\Gamma}\|c_\gamma\|^2_{C^{1,1}(B_{1/4})} + \sup_{B_1}(\partial_e u)^2 + \sup_{\mathbb{R}^n}u^2\right).$$

This yields the desired estimate (3.33) in case $u \in C^\infty(B_1)\cap L^\infty(\mathbb{R}^n)$.

Step 4: Finally, let us prove that the same estimate holds for any viscosity solution u of (3.31). Thanks to the results in [92], for any viscosity solution u of (3.31) there exists a sequence $u^{(k)} \in C^\infty(B_1)\cap L^\infty(\mathbb{R}^n)$ of solutions to the same class of equations converging to u locally uniformly in B_1. Moreover, due to [52], the solutions $u^{(k)}$ satisfy a Lipschitz regularity estimate. Thanks to this, and since we already proved (3.33) for C^∞-solutions, we obtain

$$\begin{aligned}\sup_{B_{1/8}} -\partial^2_{ee}u^{(k)} &\leq C_1\left(\sup_{\gamma\in\Gamma^{(k)}}\|c_\gamma\|_{C^{1,1}(B_1)} + \|\partial_e u^{(k)}\|_{L^\infty(B_1)} + \|u^{(k)}\|_{L^\infty(\mathbb{R}^n)}\right)\\ &\leq C_2\left(\sup_{\gamma\in\Gamma}\|c_\gamma\|_{C^{1,1}(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right),\end{aligned}$$

where $C_1, C_2 > 0$ do not depend on k. Therefore, we can take the limit $k\to\infty$, and deduce the desired result. □

Remark 3.4.3. The semiconvexity estimate (3.32) is robust with respect to the limit $s \nearrow 1$, i.e., if we choose $L \in \mathcal{L}_s((1-s)\lambda,(1-s)\Lambda;1)$ and $s \geq s_0$ for some $s_0 \in (0,1)$, then the constant C will depend only on n, s_0, λ, Λ.

Remark 3.4.4. Instead of (C^2), our proof of the semiconvexity estimate remains true under

$$\int_{\mathbb{R}^n\setminus B_{1/8}}|D^2K(y)|\,\mathrm{d}y \leq \Lambda. \tag{3.38}$$

3.5 Key estimates in terms of difference quotients

Recall that the key estimates which were established in the previous section all require some a priori smoothness assumption on u, such as $u \in C^{2+2s+\varepsilon}_{loc}$ in order to make sense of expressions like $L(\partial_e u)$ and $L(\partial^2_{ee}u)$. This way, the Bernstein technique cannot be used in order to establish smoothness of solutions, but only to prove estimates for solutions that are already known to be smooth.

Moreover, in many applications, as for example the obstacle problem, solutions are known not to possess the required $C^{2+2s+\varepsilon}$ regularity. In order to apply our technique to such situations, in this section we establish Bernstein key estimates for difference quotients. Let us mention that this idea was also announced in a version of a preprint of [33] (see Section 3 in [32]).

For $h \in \mathbb{R}^n$, and $u \in C(\mathbb{R}^n)$ let us define

$$u_h(x) := \int_0^1 u(x + th)\,\mathrm{d}t.$$

Moreover, we introduce difference quotients

$$D_h u(x) := \frac{u(x+h) - u(x)}{|h|} = \partial_e(u_h)(x),$$

where $e = h/|h|$. Clearly, by the fundamental theorem of calculus, the following identity holds once $u \in C^{0,1}(B_1(x))$:

$$D_h u(x) = \int_0^1 \partial_e u(x + th)\,\mathrm{d}t = (\partial_e u)_h(x).$$

Moreover, note that

$$D_{-h} D_h u(x) = \frac{2u(x) - u(x+h) - u(x-h)}{|h|^2}.$$

Remark 3.5.1. Note that we have the following integration by parts identity for difference quotients, whenever f, g are such that the integrals below are well-defined:

$$\int_{\mathbb{R}^n} D_h f(x) g(x)\,\mathrm{d}x = -\int_{\mathbb{R}^n} f(x) D_{-h} g(x)\,\mathrm{d}x. \tag{3.39}$$

3.5.1 Key estimate for non-smooth functions

We claim the following analogues of Theorem 3.1.6 and Theorem 3.1.8 for difference quotients:

Proof of Lemma 3.1.9. The proofs of (3.11) and (3.12) go in the exact same way as the proofs of Theorem 3.1.6 and Theorem 3.1.8, respectively, upon replacing u and v by u_h and v_h, respectively, and $\partial_e u$ and $\partial_e v$ by $D_h u$ and $D_h v$, respectively. Moreover, in order to prove (3.11), we need the following interpolation estimate, reminiscent of Lemma 3.3.4:

$$(D_h u(x))^2 \le \delta^{2s} B(D_h u, D_h u)(x) + c\delta^{2s-2} B(u_h, u_h)(x). \tag{3.40}$$

A similar estimate is required for the proof of (3.12).

The proof of the discrete interpolation estimate (3.40) also goes as before, however, the following computation has to be employed: if we denote $e = h/|h|$ then

$$\begin{aligned} \int_{\mathbb{R}^n} D_h u(y) K_\delta(x-y)\,\mathrm{d}y &= \int_{\mathbb{R}^n} \partial_e(u_h(y) - u_h(x)) K_\delta(x-y)\,\mathrm{d}y \\ &= -\int_{\mathbb{R}^n} (u_h(y) - u_h(x)) \partial_e K_\delta(x-y)\,\mathrm{d}y, \end{aligned} \tag{3.41}$$

using $D_h u = \partial_e u_h$ and integrating by parts. Having at hand the discrete interpolation estimate (3.40), the terms involving K_1 can be treated exactly as before. For K_2, we also proceed as in Step 1 of the proof of Theorem 3.1.6, employing a similar computation as (3.41) with respect to K_2 instead of K_δ. □

As in the non-discrete case, it is possible to use (3.12) in order to get a key estimate that is suitable for the application to second order estimates.

Corollary 3.5.2. *Let $s \in (s_0, 1)$, with $s_0 > 0$, and $L \in \mathcal{L}_s(\lambda, \Lambda; 1)$. Let $\eta \in C_c^{1,1}(B_1)$ be such that $\eta \equiv 1$ in $B_{1/2}$ and $0 \le \eta \le 1$. Moreover, let $\bar\eta \in C_c^{1,1}(B_{1/4})$ be such that $\bar\eta \equiv 1$ in $B_{1/8}$ and $0 \le \bar\eta \le 1$. Let $|h| \le 1/8$. Then, there exists $\sigma_0 = \sigma_0(n, s_0, \Lambda/\lambda, \|\bar\eta\|_{C^{1,1}(\mathbb{R}^n)}) > 0$ such that for every $\sigma \ge \sigma_0$ and $u \in C^{2s+\varepsilon}_{loc}(B_1)$ the following estimate holds true for $\tilde u := \eta u$*

$$\begin{aligned} &L(\bar\eta^2 (D_{-h} D_h \tilde u)_+^2 + \sigma (D_h \tilde u_{-h})^2) \\ &\qquad \le 2\bar\eta^2 L(D_{-h} D_h \tilde u)(D_{-h} D_h \tilde u)_+ + 2\sigma L(D_h \tilde u_{-h}) D_h \tilde u_{-h}. \end{aligned} \tag{3.42}$$

Proof. The proof follows by application of (3.12) with $-h$ and $v = D_h \tilde u$ and using

$$D_h(u_{-h}) = \frac{1}{h} \int_0^1 (u(\cdot - th + h) - u(\cdot - th)) \,\mathrm{d}t = (D_h u)_{-h}.$$

□

Finally, we observe that the key estimates (3.11) and (3.12) can also be obtained for Hölder difference quotients defined as

$$D_h^\alpha u(x) = \frac{u(x+h) - u(x)}{|h|^\alpha}, \quad \alpha \in (0,1).$$

Corollary 3.5.3. *Let $s \in (s_0, 1)$, with $s_0 > 0$, and $L \in \mathcal{L}_s(\lambda, \Lambda; 1)$. Let $\eta \in C_c^{1,1}(B_1)$ be such that $\eta \equiv 1$ in $B_{1/2}$ and $0 \le \eta \le 1$. Let $\alpha \in (0,1)$ and $|h| \le 1/8$. Then, there exists $\sigma_0 = \sigma_0(n, s_0, \Lambda/\lambda, \|\eta\|_{C^{1,1}(\mathbb{R}^n)}) > 0$ such that for every $\sigma \ge \sigma_0$ and $u, v \in C^{2s+\varepsilon}_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$:*

$$L(\eta^2 (D_h^\alpha u)^2 + \sigma u_h^2) \le 2\eta^2 L(D_h^\alpha u) D_h^\alpha u + 2\sigma L(u_h) u_h, \tag{3.43}$$

$$L(\eta^2 (D_h^\alpha v)_+^2 + \sigma v_h^2) \le 2\eta^2 L(D_h^\alpha v)(D_h^\alpha v)_+ + 2\sigma L(v_h) v_h. \tag{3.44}$$

Proof. We only explain how to prove (3.43), since the proof of (3.44) can be proved in the same way. We multiply (3.11) on both sides by $|h|^{2-2\alpha}$. Then, for any $\sigma \ge \sigma_0$

$$L(\eta^2 (D_h^\alpha u)^2 + |h|^{2-2\alpha} \sigma u_h^2) \le 2\eta^2 L(D_h^\alpha u) D_h^\alpha u + 2|h|^{2-2\alpha} \sigma L(u_h) u_h.$$

In particular, we obtain (3.43) for any $\sigma \ge \sigma_0 \ge \sigma_0 |h|^{2-2\alpha}$. □

3.5.2 Improved key estimate for Lipschitz continuous functions

Moreover, we can prove the following key estimate which produces slightly different averages on the lower order terms. The price we have to pay is that we need to assume certain smoothness of u:

Lemma 3.5.4. *Let $s \in (s_0, 1)$, with $s_0 > 0$, and $L \in \mathcal{L}_s(\lambda, \Lambda; 1)$. Let $\eta \in C_c^{1,1}(B_1)$ be such that $\eta \ge 0$. Let $|h| \le 1/8$. Then, there exists $\sigma_0 = \sigma_0(n, s_0, \Lambda/\lambda, \|\eta\|_{C^{1,1}(\mathbb{R}^n)}) > 0$ such that for every $\sigma \ge \sigma_0$ and $u, v \in C^{2s+\varepsilon}(B_{1/2}) \cap C^{0,1}(B_{1/2}) \cap L^\infty(\mathbb{R}^n)$, the following estimate holds true in $B_{1/4}$:*

$$L(\eta^2 (D_h u)^2 + \sigma [u^2]_h) \le 2\eta^2 L(D_h u) D_h u + 2\sigma [L(u) u]_h, \tag{3.45}$$

$$L(\eta^2 (D_h v)_+^2 + \sigma [v^2]_h) \le 2\eta^2 L(D_h v)(D_h v)_+ + 2\sigma [L(v) v]_h. \tag{3.46}$$

Proof. The proof goes exactly as the proof of Lemma 3.1.9. The only difference is how we integrate by parts with difference quotients. This is contained in the following identity, which replaces (3.41). We claim that, denoting $e = h/|h|$,

$$\int_{\mathbb{R}^n} D_h v(y) J(x-y)\,\mathrm{d}y = \left(\int_{\mathbb{R}^n} (v(\cdot) - v(y))\, \partial_e J(\cdot - y)\,\mathrm{d}y \right)_h (x), \quad x \in B_{1/4}, \tag{3.47}$$

holds true for $J = K_2$ (as in Theorem 3.1.6) and $J = K_\delta$ (as in Lemma 3.3.4), where v is Lipschitz continuous in $B_{1/2}$ and bounded. Let us first verify (3.47) for globally Lipschitz continuous v:

$$\begin{aligned}
\int_{\mathbb{R}^n} D_h v(y) J(x-y)\,\mathrm{d}y &= \int_{\mathbb{R}^n} \left[\int_0^1 \partial_e v(y+th)\,\mathrm{d}t \right] J(x-y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \left[\int_0^1 \partial_e \left(v(y+th) - v(x+th) \right)\,\mathrm{d}t \right] J(x-y)\,\mathrm{d}y \\
&= \int_0^1 \int_{\mathbb{R}^n} \partial_e \left(v(y+th) - v(x+th) \right) J(x-y)\,\mathrm{d}y\,\mathrm{d}t \\
&= -\int_0^1 \int_{\mathbb{R}^n} \left(v(y+th) - v(x+th) \right) \partial_e J(x-y)\,\mathrm{d}y\,\mathrm{d}t \\
&= -\int_0^1 \int_{\mathbb{R}^n} \left(v(y) - v(x+th) \right) \partial_e J((x+th)-y)\,\mathrm{d}y\,\mathrm{d}t \\
&= \left(\int_{\mathbb{R}^n} (v(\cdot) - v(y))\, \partial_e J(\cdot - y)\,\mathrm{d}y \right)_h (x).
\end{aligned} \tag{3.48}$$

Note that we used Lipschitz continuity of v in the first step. Moreover, we used $L([w]_h) = [Lw]_h$, which is a direct consequence of Fubini's theorem.

We remark that in case $J = K_\delta$, it suffices to have Lipschitz continuity of v in $B_{1/2}$ in order to carry out the computation in (3.48), because $\operatorname{supp}(K_\delta) \subset B_\delta(0)$ and $\delta > 0$ can be chosen to satisfy $\delta < 1/8$ in the application. This yields the following discrete interpolation estimate:

$$(D_h u(x))^2 \le \delta^{2s} B(D_h u, D_h u)(x) + c\delta^{2s-2} [B(u,u)]_h(x), \tag{3.49}$$

which allows us to treat all the terms containing K_1.

In order to prove (3.47) with $J = K_2$, we approximate any function $v \in L^\infty(\mathbb{R}^n)$ that is Lipschitz continuous in $B_{1/2}$ by a sequence of globally Lipschitz continuous functions (v_k), satisfying $\|v_k\|_{L^\infty(\mathbb{R}^n)} \le 2\|v\|_{L^\infty(\mathbb{R}^n)}$, $v_k \equiv v$ in $B_{1/4}$, and converging to v almost everywhere, as $k \to \infty$. Then, by dominated convergence, for almost every $x \in \mathbb{R}^n$:

$$\begin{aligned}
\int_{\mathbb{R}^n} D_h v(y) K_2(x-y)\,\mathrm{d}y &= \lim_{k\to\infty} \int_{\mathbb{R}^n} D_h v_k(y) K_2(x-y)\,\mathrm{d}y \\
&= \lim_{k\to\infty} \left(\int_{\mathbb{R}^n} (v_k(y) - v_k(\cdot))\, \partial_e K_2(\cdot - y)\,\mathrm{d}y \right)_h (x) \\
&= \left(\int_{\mathbb{R}^n} (v(y) - v(\cdot))\, \partial_e K_2(\cdot - y)\,\mathrm{d}y \right)_h (x),
\end{aligned}$$

where we used that K_2 and $|\nabla K_2|$ are integrable by properties (iv), (v), and (vi) in Lemma 3.3.1, and that $D_h v_k \to D_h v$ a.e. and $v_k(y) - v_k(x+th) \to v(y) - v(x+th)$ for a.e. $y \in \mathbb{R}^n$ and every $x \in B_{1/4}$. This proves (3.47) for $J = K_2$ and allows us to treat all terms involving K_2 in the proof of the key estimate. This concludes the proof. □

Finally, we explain how to obtain second order estimates from (3.46):

Corollary 3.5.5. *Let $s \in (s_0, 1)$, with $s_0 > 0$, and $L \in \mathcal{L}_s(\lambda, \Lambda; 1)$. Let $\eta \in C_c^{1,1}(B_1)$ be such that $\eta \equiv 1$ in $B_{1/2}$ and $0 \le \eta \le 1$. Moreover, let $\bar\eta \in C_c^{1,1}(B_{1/4})$ be such that $\bar\eta \equiv 1$ in $B_{1/8}$ and $0 \le \bar\eta \le 1$. Let $|h| \le 1/8$. Then, there exists $\sigma_0 = \sigma_0(n, s_0, \Lambda/\lambda, \|\bar\eta\|_{C^{1,1}(\mathbb{R}^n)}) > 0$ such that for every $\sigma \ge \sigma_0$ and $u \in C^{2s+\varepsilon}(B_{1/2}) \cap C^{0,1}(B_{1/2}) \cap L^\infty(B_1)$, the following estimate holds true in $B_{1/4}$ for $\tilde u := \eta u$:*

$$\begin{aligned} &L(\bar\eta^2 (D_{-h} D_h \tilde u)_+^2 + \sigma[(D_h \tilde u)^2]_{-h}) \\ &\qquad \le 2\bar\eta^2 L(D_{-h} D_h \tilde u)(D_{-h} D_h \tilde u)_+ + 2\sigma[L(D_h \tilde u) D_h \tilde u]_{-h}. \end{aligned} \tag{3.50}$$

Proof. The proof follows by application of (3.46) with $-h$ and $v = D_h \tilde u$. □

As in the previous section, we can also obtain an improved key inequality for Hölder difference quotients:

Corollary 3.5.6. *Let $s \in (s_0, 1)$, with $s_0 > 0$, and $L \in \mathcal{L}_s(\lambda, \Lambda; 1)$. Let $\eta \in C_c^{1,1}(B_1)$ be such that $\eta \equiv 1$ in $B_{1/2}$ and $0 \le \eta \le 1$. Let $\alpha \in (0,1)$ and $|h| \le 1/8$. Then, there exists $\sigma_0 = \sigma_0(n, s_0, \Lambda/\lambda, \|\eta\|_{C^{1,1}(\mathbb{R}^n)}) > 0$ such that for every $\sigma \ge \sigma_0$ and $u, v \in C^{2s+\varepsilon}(B_{1/2}) \cap C^{0,1}(B_{1/2}) \cap L^\infty(\mathbb{R}^n)$, the following estimate holds true in $B_{1/4}$:*

$$L(\eta^2 (D_h^\alpha u)^2 + \sigma[u^2]_h) \le 2\eta^2 L(D_h^\alpha u) D_h^\alpha u + 2\sigma[L(u)u]_h, \tag{3.51}$$

$$L(\eta^2 (D_h^\alpha v)_+^2 + \sigma[v^2]_h) \le 2\eta^2 L(D_h^\alpha v)(D_h^\alpha v)_+ + 2\sigma[L(v)v]_h. \tag{3.52}$$

Proof. The proof goes exactly as the proof of Corollary 3.5.3, but starting with (3.45) and (3.46). □

3.6 Application to the obstacle problem

In this section we apply the Bernstein technique to the obstacle problem (3.3) with $L \in \mathcal{L}_s(\lambda, \Lambda; 1)$ (see Definition 3.2.1) and with ϕ being a sufficiently smooth function.

First, we establish semiconvexity estimates for solutions to (3.3) in Subsection 3.6.1. In Subsection 3.6.2 we prove that blow-ups are convex (see Theorem 3.1.5). This result is crucial in the proof of our main results Theorem 3.1.4 and Theorem 3.1.3, which will be carried out in Subsection 3.6.3.

Let us start our discussion by several remarks.

Remark 3.6.1. It was proved in [39, Theorem 5.1] that any solution u to the obstacle problem (3.3) belongs to C^β whenever $\phi \in C^\beta$ for some $\beta < \max\{1+\varepsilon, 2s+\varepsilon\}$. In particular, if $\beta > 2s$, we may assume that u is a solution to (3.3) in the classical sense.

Remark 3.6.2. Let us also point out that once $\phi \in C^\beta$ for some $\beta > 2s$, we can rewrite (3.3) as an obstacle problem with a zero obstacle and an inhomogeneity, as follows

$$\min\{Lv - f, v\} = 0 \quad \text{in } B_1. \tag{3.53}$$

This can be achieved by extending ϕ in a smooth way to $\mathbb{R}^n$ and defining $f = L\phi \in C^{\beta-2s}$ and $v = u - \phi$ (as long as $\beta - 2s$ is not an integer). Sometimes it will be more convenient to work with the formulation (3.53) instead of (3.3).

3.6.1 Semiconvexity estimates

Using the Bernstein technique for difference quotients (see Section 3.5), we prove the semiconvexity estimate in Theorem 3.1.2. Note that we will not apply these results in the proof of optimal regularity for solutions and regularity of the free boundary. However, we consider both, the result in itself and its proof using Bernstein technique, of independent interest.

In order to apply the Bernstein technique, we need to be able to evaluate Lu and $LD_h u$ in a pointwise sense. We rely on Theorem 5.1 in [39], which states that solutions to the obstacle problem are classical once the data is smooth enough.

Proof of Theorem 3.1.2. Thanks to Remark 3.6.2, we may assume that u is a solution of (3.53), with $f = L\phi$.
Let us denote $\tilde{u} = \eta u$, where $\eta \in C_c^\infty(B_1)$, $\eta \equiv 1$ in $B_{1/2}$ and $0 \le \eta \le 1$. Let $0 < |h| < 1/16$.

Step 1: We estimate the quantity $[L(D_h\tilde{u})D_h\tilde{u}]_{-h}$ in $B_{1/4-|h|}$. Let us first prove the following claim: For all $x \in B_{1/4}$,

$$L(D_h u)(x) D_h u(x) \le \|\nabla f\|_{L^\infty(B_1)} |D_h u(x)|. \tag{3.54}$$

To prove (3.54), we first assume that $u(x) > 0$. We distinguish between two cases:
Case 1: $u(x+h) > 0$. Then, $L(D_h u)(x) = D_h f(x)$, and therefore

$$L(D_h u)(x) D_h u(x) = D_h f(x) D_h u(x) \le \|\nabla f\|_{L^\infty(B_1)} |D_h u(x)|.$$

Case 2: $u(x+h) = 0$. Then,

$$D_h u(x) = -\frac{u(x)}{h} < 0, \quad L(D_h u)(x) = \frac{Lu(x+h) - f(x)}{h} \ge D_h f(x).$$

Thus,

$$L(D_h u)(x) D_h u(x) \le D_h f(x) D_h u(x) \le \|\nabla f\|_{L^\infty(B_1)} |D_h u(x)|.$$

On the other hand, let us now assume that $u(x) = 0$. We distinguish between two cases:
Case 1: $u(x+h) = 0$. Then, $D_h u(x) = 0$, and therefore

$$L(D_h u)(x) D_h u(x) = 0.$$

Case 2: $u(x+h) > 0$. Then,

$$D_h u(x) = \frac{u(x+h)}{h} > 0, \quad L(D_h u)(x) = \frac{f(x+h) - Lu(x)}{h} \le D_h f(x).$$

Thus,

$$L(D_h u)(x) D_h u(x) \le D_h f(x) D_h u(x) \le \|\nabla f\|_{L^\infty(B_1)} |D_h u(x)|.$$

All in all, we get for any $x \in B_1$:

$$L(D_h u)(x) D_h u(x) \le D_h f(x) D_h u(x) \le \|\nabla f\|_{L^\infty(B_1)} |D_h u(x)|,$$

which yields (3.54), as desired.
Now let us turn to estimating $[L(D_h\tilde{u})D_h\tilde{u}]_{-h}$ in $B_{1/4-|h|}$. For this, we note that for $x \in B_{1/4}$, using (C^1):

$$
\begin{aligned}
|L(D_h[(1-\eta)u](x)| &= \left| \int_{\mathbb{R}^n \setminus B_{\frac{1}{4}-|h|}(x)} D_h[(1-\eta)u](y) K(x-y) \, \mathrm{d}y \right| \\
&= \left| \int_{\mathbb{R}^n \setminus B_{\frac{1}{4}-|h|}(x)} (1-\eta)u(y) D_{-h}K(x-y) \, \mathrm{d}y \right| \\
&\le C\|u\|_{L^\infty(\mathbb{R}^n)}.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
L(D_h\tilde{u})D_h\tilde{u}(x) &= L(D_h u)D_h u - L(D_h[1-\eta]u)D_h u \\
&\le C\left(\|\nabla f\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right)|D_h u(x)|
\end{aligned}
$$

Using interior Lipschitz estimates for solutions of (3.53) (see [39, Theorem 5.1]), we obtain the following estimate in $B_{1/4-|h|}$:

$$
\begin{aligned}
[L(D_h\tilde{u})D_h\tilde{u}]_{-h} &\le C\left(\|\nabla f\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right)(\|f\|_{C^{0,1}(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}) \\
&\le C(\|f\|^2_{C^{0,1}(B_1)} + \|u\|^2_{L^\infty(\mathbb{R}^n)}).
\end{aligned}
\tag{3.55}
$$

Step 2: We give an estimate for $L(D_{-h}D_h\tilde{u})(x)(D_{-h}D_h\tilde{u}(x))_+$ in $B_{1/4}$: First of all, let us assume that $u(x) > 0$. In this case

$$
\begin{aligned}
L(D_{-h}D_h u)(x) &= \frac{2Lu(x) - Lu(x+h) - Lu(x-h)}{h^2} \\
&= \frac{2f(x) - Lu(x+h) - Lu(x-h)}{h^2} \le D_{-h}D_h f(x).
\end{aligned}
$$

Moreover, since in $\{u = 0\}$, it holds $D_{-h}D_h u \le 0$, we obtain for $x \in B_1$:

$$
L(D_{-h}D_h u)(x)(D_{-h}D_h u(x))_+ \le \|D^2 f\|_{L^\infty(B_1)}(D_{-h}D_h u(x))_+. \tag{3.56}
$$

Next, note that, by a similar argument as in Step 1, but using (C^2) instead of (C^1), we obtain for $x \in B_{1/4-2|h|}$:

$$
|L(D_{-h}D_h[(1-\eta)u])(x)| \le C\|u\|_{L^\infty(\mathbb{R}^n)}. \tag{3.57}
$$

Therefore, using (3.56) and (3.57), we have the following estimate:

$$
\begin{aligned}
&L(D_{-h}D_h\tilde{u})(D_{-h}D_h\tilde{u})_+(x) \\
&\quad = L(D_{-h}D_h u)(D_{-h}D_h u(x))_+ - L(D_{-h}D_h[(1-\eta)u])(D_{-h}D_h u(x))_+ \\
&\quad \le C\left(\|D^2 f\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right)(D_{-h}D_h u(x))_+
\end{aligned}
\tag{3.58}
$$

Step 3: Finally, it remains to apply Corollary 3.5.5 and to combine it with the estimates (3.55) and (3.58) we obtained in the previous steps. Note that we can apply Corollary 3.5.5,

since we know that u is Lipschitz continuous in $B_{1/2}$.
Let $\bar{\eta} \in C_c^\infty(B_{1/4})$ be such that $\bar{\eta} \equiv 1$ in $B_{1/8}$ and $0 \le \bar{\eta} \le 1$. This yields the following estimate for every $x \in B_{1/4}$, if $|h|$ is small enough:

$$\begin{aligned}
&L(\bar{\eta}^2 (D_{-h} D_h \tilde{u})_+^2 + \sigma[(D_h \tilde{u})^2]_{-h})(x) \\
&\quad \le 2\bar{\eta}^2(x) L(D_{-h} D_h \tilde{u})(x) (D_{-h} D_h \tilde{u})_+(x) + 2\sigma [L(D_h \tilde{u}) D_h \tilde{u}]_{-h}(x) \\
&\quad \le C \bar{\eta}^2(x) \left(\|D^2 f\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)} \right) (D_{-h} D_h u(x))_+ + C(\|f\|^2_{C^{0,1}(B_1)} + \|u\|^2_{L^\infty(\mathbb{R}^n)}).
\end{aligned}$$

Consequently, by the maximum principle (see Corollary 5.2 in [164]), we have

$$\begin{aligned}
\sup_{B_{1/4}} \left(\bar{\eta}^2 (D_{-h} D_h u)_+^2 \right) &\le \sup_{B_{1/4}} \left(\bar{\eta}^2 (D_{-h} D_h \tilde{u})_+^2 + \sigma[(D_h \tilde{u})^2]_{-h} \right) \\
&\le C \sup_{B_{1/4}} \left(\bar{\eta}^2 \left(\|D^2 f\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)} \right) (D_{-h} D_h u)_+ \right) \\
&\quad + C(\|f\|^2_{C^{0,1}(B_1)} + \|u\|^2_{L^\infty(\mathbb{R}^n)}) \\
&\quad + C \sup_{\mathbb{R}^n \setminus B_{1/4}} \left(\bar{\eta}^2 (D_{-h} D_h \tilde{u})_+^2 + \sigma[(D_h \tilde{u})^2]_{-h} \right) \\
&\le \frac{1}{2} \sup_{B_{1/4}} \left(\bar{\eta}^2 (D_{-h} D_h u)_+^2 \right) + C(\|f\|^2_{C^{1,1}(B_1)} + \|u\|^2_{L^\infty(\mathbb{R}^n)}) \\
&\quad + C \sup_{\mathbb{R}^n \setminus B_{1/4}} [(D_h \tilde{u})^2]_{-h},
\end{aligned} \tag{3.59}$$

where we used Young's inequality. By absorption of the first term on the right hand side, as well as

$$\sup_{\mathbb{R}^n \setminus B_{1/4}} [(D_h \tilde{u})^2]_{-h} \le \sup_{B_{1/2}} (D_h u)^2 \le C(\|f\|^2_{C^{0,1}(B_1)} + \|u\|^2_{L^\infty(\mathbb{R}^n)}),$$

which is again due to the interior Lipschitz estimates in [39, Theorem 5.1], we end up with

$$\sup_{B_{1/8}} \left((D_{-h} D_h u)_+^2 \right) \le \sup_{B_{1/4}} \left(\bar{\eta}^2 (D_{-h} D_h u)_+^2 \right) \le C(\|f\|^2_{C^{1,1}(B_1)} + \|u\|^2_{L^\infty(\mathbb{R}^n)}). \tag{3.60}$$

Since this estimate is uniform in h, we can pass to the limit $h \to 0$ and deduce

$$\sup_{B_{1/8}} (-\partial^2_{ee} u)_+ \le C(\|f\|_{C^{1,1}(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}).$$

This concludes the proof. □

Remark 3.6.3. The semiconvexity estimate we just proved is robust with respect to the limit $s \nearrow 1$, i.e., if we choose $L \in \mathcal{L}_s((1-s)\lambda, (1-s)\Lambda; 1)$ and $s \ge s_0$ for some $s_0 \in (0,1)$, then the constant C will depend only on n, s_0, λ, Λ

Remark 3.6.4. Note that our proof of the semiconvexity estimate remains true under (3.38) instead of (C^2).

3.6.2 Convexity of blow-ups

In this section, we show how to apply the Bernstein technique in order to prove Theorem 3.1.5, which states that blow-ups for the nonlocal obstacle problem are necessarily convex. As explained before, this result is a central ingredient in our proof of Theorem 3.1.4 and Theorem 3.1.3.

Proof of Theorem 3.1.5. Let us take two cutoff functions $\eta \in C_c^\infty(B_2)$ and $\bar\eta \in C_c^\infty(B_{1/2})$ with $\eta, \bar\eta \ge 0$ and $\eta \equiv 1$ in B_1, $\bar\eta \equiv 1$ in $B_{1/4}$. Then, by Corollary 3.5.5 applied with $-h$, we obtain for $x \in B_{1/2}$:

$$\begin{aligned} &L(\bar\eta^2 (D_{-h}D_h(\eta u_0))_+^2 + \sigma[(D_h(\eta u_0))^2]_{-h})(x) \\ &\qquad \le 2\bar\eta^2 L(D_{-h}D_h(\eta u_0))(D_{-h}D_h(\eta u_0))_+(x) + 2\sigma[L(D_h(\eta u_0))D_h(\eta u_0)]_{-h}(x). \end{aligned}$$

Since for $|h|$ small enough and $x \in B_{1/2}$

$$D_{-h}D_h(u_0)(x) = \frac{2u_0(x) - u_0(x+h) - u_0(x-h)}{h^2} = \frac{\frac{u_0(x)-u_0(x+h)}{|h|} + \frac{u_0(x)-u_0(x-h)}{|h|}}{|h|},$$

using (3.6) for $x \in B_{1/2} \cap \{u_0 > 0\}$, and $D_{-h}D_h u_0 \le 0$ in $B_{1/2} \cap \{u_0 = 0\}$, we obtain

$$L(D_{-h}D_h u_0)(D_{-h}D_h u_0)_+(x) \le 0, \quad x \in B_{1/2}.$$

Moreover, note that after integrating by parts and using (3.5) and (C^1), $|D_h[(1-\eta)u_0]| \le C(|u_0| + |\nabla u_0|)$ and $D_h K(x-y) \le c|x-y|^{-n-1-2s} \le C|y|^{-n-1-2s}$ for $x \in B_{1/8}$ and $y \notin B_{1/4-2|h|}$. Hence,

$$\begin{aligned} |L(D_{-h}D_h[(1-\eta)u_0])| &= \left| \int_{\mathbb{R}^n \setminus B_{1/4-2|h|}} -D_{-h}D_h[(1-\eta)u_0](y)K(x-y)\,\mathrm{d}y \right| \\ &= \left| \int_{\mathbb{R}^n \setminus B_{1/4-2|h|}} D_h[(1-\eta)u_0](y)D_hK(x-y)\,\mathrm{d}y \right| \\ &\le C \int_{\mathbb{R}^n \setminus B_{1/4-2|h|}} (\|u_0\|_{L^\infty(B_3)} + |y|^{s+\alpha})|x-y|^{-n-1-2s}\,\mathrm{d}y \\ &\le C(\|u_0\|_{L^\infty(B_3)} + 1). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} &L(D_{-h}D_h(\eta u_0))(D_{-h}D_h(\eta u_0))_+(x) \\ &\quad = (L(D_{-h}D_h u_0) - L(D_{-h}D_h[(1-\eta)u_0]))\,(D_{-h}D_h(\eta u_0))_+(x) \\ &\quad \le C(\|u_0\|_{L^\infty(B_3)} + 1)(D_{-h}D_h(\eta u_0)(x))_+. \end{aligned} \tag{3.61}$$

Moreover, we obtain

$$L(D_h u_0)D_h u_0(x) \le 0 \quad \text{in} \quad B_{3/4}$$

by using the same argument as in Step 1 of the proof of Theorem 3.1.2 together with (3.6). Indeed, if $u_0(x), u_0(x+h) > 0$, (3.6) applied with x and with $x+h$ implies that $L(D_h u_0) = 0$,

and if $u_0(x) = u_0(x+h) = 0$, then $D_h u_0(x) = 0$. Moreover, if $u_0(x) > u_0(x+h) = 0$, then $L(D_h u_0) \ge 0$ and $D_h u_0 < 0$. The case $u_0(x+h) > u_0(x) = 0$ follows by changing roles of x and $x+h$.
Therefore, we deduce

$$\begin{aligned} L(D_h(\eta u_0))D_h(\eta u_0)(x) &= \left(L(D_h u_0) - L(D_h[(1-\eta)u_0])\right) D_h u_0(x) \\ &\le C(\|u_0\|_{L^\infty(B_3)} + 1)|D_h u_0(x)|, \end{aligned} \tag{3.62}$$

where we used (3.5) again as above to estimate $-L(D_h[(1-\eta)u_0]) \le C(\|u_0\|_{L^\infty(B_3)} + 1)$. Altogether, by combining (3.61) and (3.62) we obtain for any $x \in B_{1/4}$

$$\begin{aligned} &L(\bar\eta^2(D_{-h}D_h(\eta u_0))_+^2 + \sigma[(D_h(\eta u_0))^2]_{-h})(x) \\ &\quad \le C\bar\eta^2(\|u_0\|_{L^\infty(B_3)} + 1)(D_{-h}D_h(\eta u_0)(x))_+ + C(\|u_0\|_{L^\infty(B_3)} + 1)|D_h u_0(x)|. \end{aligned}$$

Consequently, by the maximum principle (see Corollary 5.2 in [164]), the definitions of $\eta, \bar\eta$, and an application of Young's inequality (as in (3.59)), we obtain

$$\begin{aligned} \sup_{B_{1/4}}(D_{-h}D_h(\eta u_0))_+^2 &\le C\left(\|u_0\|_{L^\infty(B_3)}^2 + \sup_{\mathbb{R}^n\setminus B_{1/2}}[(D_h(\eta u_0))^2]_{-h}\right) \\ &\le C\left(\|u_0\|_{L^\infty(B_3)}^2 + \|\nabla u_0\|_{L^\infty(B_3)}^2\right). \end{aligned}$$

In particular, upon sending $h \to 0$ and using (3.5), we have shown

$$-D^2 u_0 \le C\left(\|u_0\|_{L^\infty(B_3)} + \|\nabla u_0\|_{L^\infty(B_3)}\right) < \infty \quad \text{in } B_{1/4}.$$

Next, we observe that for any $r \ge 1$, the function

$$u_0^{(r)}(x) := \frac{u_0(rx)}{r^{1+s+\alpha}}$$

satisfies all the assumptions of the theorem, and therefore, by application of the same arguments as before, we obtain for $x \in B_{1/4}$:

$$-D^2 u_0^{(r)}(x) \le C\left(\|u_0^{(r)}\|_{L^\infty(B_3)} + \|\nabla u_0^{(r)}\|_{L^\infty(B_3)}\right) \le C\frac{\|u_0\|_{L^\infty(B_{3r})}}{r^{s+\alpha+1}} + \frac{\|\nabla u_0\|_{L^\infty(B_{3r})}}{r^{s+\alpha}} \le C,$$

where we used (3.5) and the fact that (3.5) also implies a growth control on u itself:

$$|u(x)| \le cr^{s+\alpha+1} \quad \forall x \in B_r, \quad \forall r \ge 1. \tag{3.63}$$

Consequently, for any $x \in \mathbb{R}^n$, by choosing $r = 8R|x|$ for $R > 1$, where $\tilde x = \frac{x}{8R|x|} \in B_{1/4}$:

$$D^2 u_0(x) = r^{s+\alpha-1}D^2 u_0^{(r)}(\tilde x) \le Cr^{s+\alpha-1} \le C|x|^{s+\alpha-1}R^{s+\alpha-1} \to 0 \quad \text{as } R \to \infty,$$

where we used that $s + \alpha < 1$. This implies the desired result.

Finally, notice that when K is homogeneous we can then apply the classification of blow-ups in [46, Theorem 4.1]) to obtain $u_0(x) = \kappa(x\cdot\nu)_+^{1+s}$ for some $\nu \in \mathbb{S}^{n-1}$, $\kappa \ge 0$. □

3.6.3 Optimal regularity for solutions

We next turn our attention to proving Theorem 3.1.4 and Theorem 3.1.3, i.e., optimal C^{1+s} regularity for solutions to the obstacle problem (3.53) and regularity of the free boundary near regular points. Moreover, we also prove optimal regularity estimates in the presence of non-smooth obstacles $\phi \in C^{\beta}$ with $\beta < 1+s$. Throughout this subsection, we assume that $L \in \mathcal{L}_s(\lambda, \Lambda; 1)$ and is homogeneous, i.e., (3.4) holds.

The proofs in this section follow the same overall strategy as the proof of optimal regularity for the global obstacle problem in [104] (see also [64]). However, as we explained before, we do not deduce the convexity of the blow-ups from semiconvexity estimates of solutions, but apply Theorem 3.1.5, where we proved it directly. In the sequel we will sketch the proof of our main results, following the scheme in [104] and emphasizing the main differences.

As a preparation, we need the following slightly improved version of Theorem 5.1 from [39], whose proof uses several ideas from Lemma 3.6 in [1]. For the sake of readability, we postpone it to the Appendix (see Section 3.8).

Lemma 3.6.5. *Let $s \in (0,1)$, $L \in \mathcal{L}_s(\lambda, \Lambda; 1)$. Let $f \in C^{\beta-2s}(B_1)$ for some $\beta \in (2s, 1+s)$, and let $\alpha \in (0,s)$. Let u be a viscosity solution to the obstacle problem (3.53). Then, $u \in C^{\max\{2s+\varepsilon,1+\varepsilon\}}(B_{1/2})$ and it satisfies the following estimate*

$$\|u\|_{C^{\max\{2s+\varepsilon,1+\varepsilon\}}(B_{1/2})} \le C\left([f]_{C^{\beta-2s}(B_1)} + \left\|\frac{u}{(1+|x|)^{1+s+\alpha}}\right\|_{L^\infty(\mathbb{R}^n)}\right),$$

where $C = C(n,s,\lambda,\Lambda) > 0$ is a uniform constant.

Using this, we can now give the:

Proof of Theorem 3.1.3 for $\beta < 1+s$. First of all, recall that the case $\beta < \max\{2s+\varepsilon, 1+\varepsilon\}$ was done in [39, Theorem 5.1]. In particular, we may assume $\beta > \max\{2s, 1\}$ and that $u \in C^{2s+\varepsilon}(B_1)$ for some small $\varepsilon > 0$, i.e., u is a classical solution. Our goal is to prove that, when $\beta \in (\max\{2s,1\}, 1+s)$, we have

$$|\nabla(u-\phi)(x)| \le C(\|\phi\|_{C^\beta(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)})|x-x_0|^{\beta-1} \tag{3.64}$$

for any free boundary point $x_0 \in \partial\{u > \phi\}$.
Our proof is based on the ideas of the proof of [104, Corollary 2.12]. After a normalization, we can assume without loss of generality that $x_0 = 0$ and that $\|\phi\|_{C^\beta(\mathbb{R}^n)} + \|u\|_{L^\infty(\mathbb{R}^n)} = 1$. Moreover, let us replace u by $w := v\eta := (u-\phi)\eta$ for some cutoff-function $\eta \in C_c^\infty(B_4)$ with $0 \le \eta \le 1$ and $\eta \equiv 1$ in B_3. Clearly, w satisfies

$$\min\{Lw - f, w\} = 0 \quad \text{in} \quad B_2,$$

where $f = -L\phi - L((1-\eta)v) \in C^{\beta-2s}(B_1)$ satisfies $\|f\|_{C^{\beta-2s}(B_1)} \le C$ for a uniform constant $C = C(n,s,\lambda,\Lambda,\beta) > 0$. Therefore, by [39, Theorem 5.1] (see also Lemma 3.6.5), it holds that ∇w is globally bounded by a uniform constant. Note that the regularity of f follows since by $\phi \in C^\beta(B_2)$, we have that $L\phi \in C^{\beta-2s}(B_1)$, and moreover, $L((1-\eta)v) \in C^{0,1}(B_1)$, as one can see from ($C^1$) and the identity

$$L(v(1-\eta))(x) = -\int_{\mathbb{R}^n} [v(1-\eta)](y) K(x-y)\,\mathrm{d}y. \tag{3.65}$$

Let us now begin with the proof of (3.64). We need to show that

$$|\nabla w(x)| \leq C|x|^{\mu}, \tag{3.66}$$

where $\mu = \beta - 1$ and C depends only on n, s, β, λ, Λ. Let us assume by contradiction that (3.66) does not hold. Then, we can find sequences w_k, L_k, f_k with the properties discussed before such that $0 \in \partial\{w_k > 0\}$ and

$$\sup_k \sup_{r>0} \frac{\|\nabla w_k\|_{L^\infty(B_r)}}{r^{\mu}} = \infty. \tag{3.67}$$

By Lemma 2.12 in [104], there exist sequences $r_m \to 0$ and $k_m \to 0$, as $m \to \infty$, such that

$$\widetilde{w}_m = \frac{w_{k_m}(r_m x)}{r_m \|\nabla w_{k_m}\|_{L^\infty(B_{r_m})}}, \quad \nabla \widetilde{w}_m = \frac{\nabla w_{k_m}(r_m x)}{\|\nabla w_{k_m}\|_{L^\infty(B_{r_m})}}$$

satisfy

$$\begin{aligned} \|\nabla \widetilde{w}_m\|_{L^\infty(B_1)} = 1, \quad |\nabla \widetilde{w}_m(x)| \leq C(1+|x|^{\mu}) \quad \forall x \in \mathbb{R}^n, \\ \min\{L_m \widetilde{w}_m - \widetilde{f}_m, \widetilde{w}_m\} = 0 \quad \text{in } B_{2/r_m}, \end{aligned}$$

where we write $L_m := L_{k_m}$ and define

$$\widetilde{f}_m := r_m^{2s-1} \frac{f_{k_m}(r_m x)}{\|\nabla w_{k_m}\|_{L^\infty(B_{r_m})}}.$$

Consequently, by the uniform control on $\nabla \widetilde{w}_m$, we have $\widetilde{w}_m \to \widetilde{w}_0$ locally uniformly for some limiting function $\widetilde{w}_0$ up to a subsequence. Moreover, observe that

$$L_m(D_h \widetilde{w}_m) \geq D_h \widetilde{f}_m \quad \text{in } \{\widetilde{w}_m > 0\} \cap B_{1/r_m}, \tag{3.68}$$

$$L_m(D_h \widetilde{w}_m) = D_h \widetilde{f}_m \quad \text{in } \{x : \operatorname{dist}(x, \{\widetilde{w}_m = 0\}) > |h|\} \cap B_{1/r_m}, \tag{3.69}$$

and that by the proof of Lemma 2.12 in [104] we also have

$$r_m^{-\mu} \|\nabla w_{k_m}\|_{L^\infty(B_{r_m})} \geq \frac{1}{2} \sup_k \sup_{r \geq r_m} r^{-\mu} \|\nabla w_k\|_{L^\infty(B_r)}.$$

By (3.67), we can extract a further subsequence $(l_m) \subset (k_m)$ such that for every $m \in \mathbb{N}$:

$$\|\nabla w_{l_m}\|_{L^\infty(B_{r_{l_m}})} \geq m r_{l_m}^{\mu}. \tag{3.70}$$

To simplify the presentation of the proof, let us slightly abuse notation and write again r_m and k_m instead of r_{l_m} and l_m. Now, since $f_{k_m} \in C^{\beta-2s}(B_1)$ we have for any $|h| > 0$ and $x \in B_{1/r_m}$:

$$|D_h \widetilde{f}_m(x)| \leq m^{-1} r_m^{2s-1-\mu} |D_h f_{k_m}(r_m x)| \leq c m^{-1} r_m^{\beta-1-\mu} |h|^{\beta-2s-1} \to 0, \quad \text{as } m \to \infty, \tag{3.71}$$

where we used (3.70) and $\mu = \beta - 1$. Therefore, using (3.68), (3.71), and the stability of viscosity solutions, we obtain

$$L_\infty(D_h \widetilde{w}_0) \geq 0 \quad \text{in } \{\widetilde{w}_0 > 0\},$$

where L_∞ denotes the weak limit of $(L_m)_m$. Moreover, we have $\widetilde{w}_m \in C^{2s+\varepsilon}(B_{1/r_m})$ and

$$\|\widetilde{w}_m\|_{C^{2s+\varepsilon}(B_{1/r_m})} \le C \tag{3.72}$$

for a uniform constant $C = C(n, s, \lambda, \Lambda) > 0$. This is a consequence of Lemma 3.6.5, the gradient control on $\widetilde{w}_m$ and the fact that $[\widetilde{f}_m]_{C^{\beta-2s}(B_{1/r_m})} \to 0$, as $m \to \infty$, which follows from the computation

$$|D_h^{\beta-2s} \widetilde{f}_m(x)| \le m^{-1} r_m^{2s-1-\mu} |D_h^{\beta-2s} f_{k_m}(r_m x)| \le c m^{-1} r_m^{\beta-1-\mu} \quad \forall |h| > 0.$$

By (3.72), we have uniform convergence $\widetilde{w}_m \to \widetilde{w}_0 \in C^{2s+\varepsilon}_{loc}(\mathbb{R}^n)$, and by using also (3.69) and (3.71), we can conclude

$$L_\infty(\nabla \widetilde{w}_0) = 0 \quad \text{in } \{\widetilde{w}_0 > 0\}.$$

and also

$$\|\nabla \widetilde{w}_0\|_{L^\infty(B_R)} \le c R^\mu, \quad \forall R \ge 1. \tag{3.73}$$

Now, we are in the position to apply Theorem 3.1.5. This yields convexity of $\widetilde{w}_0$, and

$$\widetilde{w}_0(x) = \kappa (x \cdot e)_+^{1+s}$$

for some $\kappa \ge 0$ and $e \in \mathbb{S}^{n-1}$. By convexity of $\widetilde{w}_0$ and $\|\nabla \widetilde{w}_m\|_{L^\infty(B_1)} = 1$, it holds that $\|\nabla \widetilde{w}_0\|_{L^\infty(B_1)} \ge 1$. Therefore $\kappa \ne 0$. However, since $\mu = \beta - 1 < s$, this is a contradiction to (3.73). Therefore, we have proved (3.66) and (3.64). Note that (3.64) implies the desired result upon combining it with interior regularity estimates. □

As in [104], the proof of the regularity of the free boundary (see Theorem 3.1.4) and the optimal regularity (see Theorem 3.1.3 in case $\beta > 1+2s$) are based on the following quantitative estimate. Note that in contrast to [104, Theorem 2.2], we do not need to assume semiconvexity of u.

Lemma 3.6.6. *Let $s \in (0,1)$, $L \in \mathcal{L}_s(\lambda, \Lambda; 1)$ satisfying (3.4). Let $\alpha \in (0,s) \cap (0, 1-s)$. Let $\eta > 0$ and assume that $u \in C^{0,1}(\mathbb{R}^n)$ satisfies*

$$\begin{aligned} u \ge 0 \ \text{in } B_{1/\eta}, \quad &\text{with } 0 \in \partial\{u > 0\}, \\ \min\{Lu - f, u\} = 0 \quad \text{in } B_{1/\eta}, \quad &\text{with } |\nabla f| \le \eta, \\ \|\nabla u\|_{L^\infty(B_R)} \le R^{s+\alpha}, \quad &\forall R \ge 1. \end{aligned}$$

Then, there exist $e \in \mathbb{S}^{n-1}$ and $\kappa \ge 0$ such that

$$\|u - \kappa (x \cdot e)_+^{1+s}\|_{C^{0,1}(B_1)} \le \varepsilon(\eta), \tag{3.74}$$

where $\varepsilon(\eta)$ is a modulus of continuity, depending only on $n, s, \lambda, \Lambda, \alpha$.
Moreover, for any $\kappa_0 > 0$ there is $\varepsilon_0 > 0$ such that if $\kappa \ge \kappa_0$ and $\varepsilon(\eta) \le \varepsilon_0$, then $\partial\{u > 0\}$ is a $C^{1,\gamma}$-graph in $B_{1/2}$ for some $\gamma > 0$ depending only on n, s, λ, Λ, α, κ_0, and

$$|u - \kappa (x \cdot e)_+^{1+s}| \le C |x|^{1+s+\gamma}, \quad |\nabla u - \nabla \kappa (x \cdot e)_+^{1+s}| \le C |x|^{s+\gamma} \quad \text{in } B_1,$$

where $C > 0$ depends only on n, s, λ, Λ, α, κ_0.

Proof. First, we prove (3.74). Let us assume by contradiction that there exists $\varepsilon > 0$ and a sequence $\eta_k \to 0$ and a sequence of operators L_k and solutions u_k satisfying the assumptions of Lemma 3.6.6 for which

$$\|u_k - \kappa(x \cdot e)_+^{1+s}\|_{C^{0,1}(B_1)} > \varepsilon \tag{3.75}$$

for any $e \in \mathbb{S}^{n-1}$ and $\kappa \geq 0$. Similar to the proof of Theorem 3.1.3 in case $\beta < 1+s$, we can show that $\|u_k\|_{C^{1+s-\varepsilon}(B_1)}$ is uniformly bounded and that u_k converges, up to a subsequence, in the C^1-norm to a limiting function u_0 satisfying the assumptions of Theorem 3.1.5. Consequently, there exist $e \in \mathbb{S}^{n-1}$ and $\kappa \geq 0$ such that $u_0(x) = \kappa(x \cdot e)_+^{1+s}$. This contradicts (3.75). Thus, we have shown (3.74). The remaining assertions follow by the same arguments as in the proof of [104, Theorem 2.2]. Note that their proof does not require semiconvexity of u. $\square$

We are now ready to establish the dichotomy between regular and degenerate free boundary points, show $C^{1,\gamma}$-regularity of the free boundary near regular points (see Theorem 3.1.4), and establish the optimal C^{1+s}-regularity of solutions (see Theorem 3.1.3 in case $\beta > 1 + 2s$).

Proof of Theorem 3.1.4 and of Theorem 3.1.3 in case $\beta > 1 + 2s$. The proof follows by the same arguments as the proof of [104, Corollary 2.16], using Lemma 3.6.6 instead of [104, Theorem 2.2]. The only difference to the proof of [104, Corollary 2.16] is that we need to use the slightly modified rescaling

$$w_k(x) := \frac{\eta}{[L\phi]_{C^{0,1}(\mathbb{R}^n)}} \frac{(u - \phi)(\eta^{-1}2^{-k}x)}{(\eta^{-1}2^{-k})^{1+s+\alpha}}, \quad k \in \mathbb{N},$$

where $\alpha \in (0, s) \cap (0, 1 - s)$ (as in Lemma 3.6.6), to guarantee that w_k solves

$$\min\{Lw_k - f_k, w_k\} = 0 \quad \text{in } B_{1/\eta}, \quad \text{with } |\nabla f_k| \leq \eta,$$

as required in the application of Lemma 3.6.6. The rest of the proof follows along the lines of the proof of [104, Corollary 2.16]. $\square$

3.7 Extensions of the Bernstein technique

In this section, we present several possible extensions of the Bernstein technique. To be more precise, in Subsection 3.7.1, we explain how to establish Bernstein key estimates for operators of the form $\partial_t + L$ and explain how to derive a priori estimates for solutions to the corresponding parabolic equation $\partial_t u + Lu = f$. Moreover, in Subsection 3.7.2, we extend the key estimates Theorem 3.1.6 and Theorem 3.1.8 to operators with nonsymmetric kernels that possess first order drift terms.

Remark 3.7.1 (Bernstein technique for convolution operators). We mention that the Bernstein technique also works for nonlocal operators with integrable kernels of the form

$$Lu(x) = \int_{\mathbb{R}^n} (u(x) - u(y))K(x - y)\,dy = u(x) - \int_{\mathbb{R}^n} u(y)K(x - y)\,dy.$$

In fact, we have the following:

(i) Assume that $K \in C^{0,1}(\mathbb{R}^n)$ satisfies for some $\Lambda > 0$

$$|\nabla K(y)| \le \Lambda K(y), \qquad \int_{\mathbb{R}^n} K(y)\,\mathrm{d}y = 1.$$

Let L be as before. Let $\eta \in C_c^\infty(B_1)$ be such that $\eta \equiv 1$ in $B_{1/2}$ and $0 \le \eta \le 1$. Then, there exists $\sigma_0 = \sigma_0(n,\Lambda) > 0$ such that for every $\sigma \ge \sigma_0$ and $u \in L^\infty(\mathbb{R}^n)$ the key estimates for difference quotients (3.11) and (3.12) hold true. Moreover, if $u \in C^{1+\varepsilon}(\mathbb{R}^n)$ the key estimates (3.8) and (3.10) hold true.

(ii) The proof of (i) follows along the lines of Step 1 in the proof of the key estimate Theorem 3.1.6 (respectively its modifications), replacing K_2 by K.

(iii) Using the key estimate for difference quotients, one can prove that solutions u to $Lu = f$ in Ω with $f \in C^{0,1}(\Omega)$ satisfy $u \in C^{0,1}_{loc}(\Omega)$. In this way, one can argue that the Bernstein technique also extends to the limiting case $s = 0$.

3.7.1 Parabolic nonlocal equations

So far, in this work we have restricted ourselves to the study of elliptic problems. In this section, we explain how the Bernstein technique can be used to study regularity estimates for parabolic equations governed by nonlocal operators. As an application, we establish semiconvexity estimates for solutions to the parabolic obstacle problem.

Theorem 3.7.2. *Let $s \in (0,1)$, $L_{K^{(t)}} \in \mathcal{L}_s(\lambda,\Lambda;1)$ for any $t \in [0,\infty)$. Let $\eta_1^{2s} \in C^{0,1}([0,1])$ be such that $\eta_1 \ge 0$. Let $\eta_2 \in C_c^{1,1}(B_1)$ be such that $\eta_2 \ge 0$. We define $\eta := \eta_1\eta_2$. Then, there exists $\sigma_0 = \sigma_0(n,s,\lambda,\Lambda,\|\eta_1^{2s}\|_{C^{0,1}([0,1])},\|\eta_2\|_{C^{1,1}(B_1)}) > 0$ such that for every $\sigma \ge \sigma_0$ and $u,v \in C^\infty((0,\infty)\times\mathbb{R}^n)$*

$$(\partial_t + L_{K^{(t)}})(\eta^2(\partial_e u)^2 + \sigma u^2) \le 2\eta^2(\partial_t + L_{K^{(t)}})(\partial_e u)\partial_e u + 2\sigma(\partial_t + L_{K^{(t)}})(u)u, \tag{3.76}$$
$$(\partial_t + L_{K^{(t)}})(\eta^2(\partial_e v)_+^2 + \sigma u^2) \le 2\eta^2(\partial_t + L_{K^{(t)}})(\partial_e v)(\partial_e v)_+ + 2\sigma(\partial_t + L_{K^{(t)}})(v)v. \tag{3.77}$$

Remark 3.7.3. By adapting the proofs in Section 3.5 to the parabolic setting, it is also possible to establish key estimates in terms of difference quotients for $(\partial_t + L_{K^{(t)}})$:

$$(\partial_t + L_{K^{(t)}})(\eta^2(D_h u)^2 + \sigma u_h^2) \le 2\eta^2(\partial_t + L_{K^{(t)}})(D_h u)D_h u + 2\sigma(\partial_t + L_{K^{(t)}})(u_h)u_h, \tag{3.78}$$
$$(\partial_t + L_{K^{(t)}})(\eta^2(D_h u)^2 + \sigma[u^2]_h) \le 2\eta^2(\partial_t + L_{K^{(t)}})(D_h u)D_h u + 2\sigma[(\partial_t + L_{K^{(t)}})(u)u]_h. \tag{3.79}$$

Proof of Theorem 3.7.2. (i) First, we prove (3.76). Let us compute

$$\begin{aligned}(\partial_t + L_{K^{(t)}})(\eta^2(\partial_e u)^2 + \sigma u^2) &= 2(\partial_t\eta_1)\eta_1\eta_2^2(\partial_e u)^2 \\ &\quad + 2\eta^2(\partial_t + L_{K^{(t)}})(\partial_e u)(\partial_e u) + 2\sigma(\partial_t + L_{K^{(t)}})(u)u \\ &\quad + L_{K^{(t)}}(\eta^2)(\partial_e u)^2 - B_{K^{(t)}}(\eta^2,(\partial_e u)^2) - \eta^2 B_{K^{(t)}}(\partial_e u,\partial_e u) - \sigma B_{K^{(t)}}(u,u).\end{aligned}$$

Therefore, it remains to prove

$$2(\partial_t\eta_1)\eta_1\eta_2^2(\partial_e u)^2 + L_{K^{(t)}}(\eta^2)(\partial_e u)^2 - B_{K^{(t)}}(\eta^2,(\partial_e u)^2) \le \eta^2 B_{K^{(t)}}(\partial_e u,\partial_e u) + \sigma B_{K^{(t)}}(u,u).$$

As in the proof of Theorem 3.1.6, let us choose $\varepsilon = \gamma\eta(x)$, where $\gamma > 0$ is as before. Therefore, by Step 1 of the proof of Theorem 3.1.6, it remains to prove

$$2(\partial_t\eta_1)\eta_1\eta_2^2(\partial_e u)^2 + L_{K_1^{(t)}}(\eta^2)(\partial_e u)^2 - B_{K_1^{(t)}}(\eta^2,(\partial_e u)^2) \le \eta^2 B_{K_1^{(t)}}(\partial_e u,\partial_e u) + \sigma B_{K_1^{(t)}}(u,u),$$

where $K_1^{(t)}$ is defined as in Lemma 3.3.1. Moreover, by carefully tracking Step 2 of the proof of Theorem 3.1.6, it becomes apparent that

$$\begin{aligned} L_{K_1^{(t)}}(\eta^2)(\partial_e u)^2 - B_{K_1^{(t)}}(\eta^2,(\partial_e u)^2) &= \eta_1^2\left(L_{K_1^{(t)}}(\eta_2^2)(\partial_e u)^2 - B_{K_1^{(t)}}(\eta_2^2,(\partial_e u)^2)\right)\\ &\le \frac{3}{4}\eta^2 B_{K_1^{(t)}}(\partial_e u,\partial_e u) + \sigma_1 B_{K_1^{(t)}}(u,u) \end{aligned}$$

for some $\sigma_1 = \sigma_1(n,s,\lambda,\Lambda,\|\eta_2\|_{C^{1,1}(\mathbb{R}^n)})$. Therefore, it remains to prove

$$2(\partial_t\eta_1)\eta_1\eta_2^2(\partial_e u)^2 \le \frac{1}{4}\eta^2 B_{K_1^{(t)}}(\partial_e u,\partial_e u) + \sigma_2 B_{K_1^{(t)}}(u,u) \tag{3.80}$$

for some $\sigma_2 = \sigma_2(n,s,\lambda,\Lambda,\|\eta_1\|_{C^{0,1}([0,1])},\|\eta_2\|_{C^{1,1}(\mathbb{R}^n)})$. Note that (3.80) is trivial once $\partial_t\eta_1(t) \le 0$. Thus, we can assume without loss of generality that $\partial_t\eta_1(t) > 0$ and choose $\delta = \left(\frac{\eta_1(t)}{8\partial_t\eta_1(t)}\right)^{\frac{1}{2s}} \wedge \frac{\varepsilon}{2}$ and apply the interpolation estimate Lemma 3.3.4. Then,

$$\begin{aligned} 2(\partial_t\eta_1)\eta_1\eta_2^2(\partial_e u)^2 &\le 2(\partial_t\eta_1)\eta_1\eta_2^2\left(\delta^{2s}B_{K_1^{(t)}}(\partial_e u,\partial_e u) + c\delta^{2s-2}B_{K_1^{(t)}}(u,u)\right)\\ &\le \frac{1}{4}\eta^2 B_{K_1^{(t)}}(\partial_e u,\partial_e u) + c(\partial_t\eta_1)^{\frac{1}{s}}\eta_1^{2-\frac{1}{s}}\eta_2^2 B_{K_1^{(t)}}(u,u)\\ &\le \frac{1}{4}\eta^2 B_{K_1^{(t)}}(\partial_e u,\partial_e u) + c\eta_2^2 B_{K_1^{(t)}}(u,u). \end{aligned}$$

Note that in the last step we used that $(\partial_t\eta_1)^{\frac{1}{s}}\eta_1^{2-\frac{1}{s}} = c(s)(\partial_t(\eta_1^{2s}))^{\frac{1}{s}}$ and η_2^2 are bounded. This establishes (3.80), as desired.

(ii): Now, let us show (3.77). The proof follows by a combination of the arguments in the proof of (3.76) and Theorem 3.1.8. In fact, it remains to prove

$$2(\partial_t\eta_1)\eta_1\eta_2^2(\partial_e v)^2 \le \frac{1}{4}\eta^2\left[B_{K_1^{(t)}}((\partial_e v)_+,(\partial_e v)_+) - 2L_{K_1^{(t)}}((\partial_e v)_-)(\partial_e v)_+\right] + \sigma_2 B_{K_1^{(t)}}(v,v) \tag{3.81}$$

for some $\sigma_2 = \sigma_2(n,s,\lambda,\Lambda,\|\eta_1\|_{C^{0,1}([0,1])},\|\eta_2\|_{C^{1,1}(\mathbb{R}^n)})$. This will be achieved by application of Lemma 3.3.8 with $\delta = \frac{1}{8}\left(\frac{\eta_1(t)}{\partial_t\eta_1(t)}\right)^{\frac{1}{2s}} \wedge \varepsilon$, where $\varepsilon > 0$ is as in the proof of Theorem 3.1.8. The rest of the proof goes as in (i). □

We also have one-sided key estimates of second order:

Corollary 3.7.4. *Let $s \in (0,1)$, $L_{K^{(t)}} \in \mathcal{L}_s(\lambda,\Lambda;1)$ for any $t \in [0,\infty)$. Let $\eta_1 \in C^\infty([0,1])$ be such that $\eta_1 \equiv 1$ in $[1/4,1]$ with $\eta_1 = 0$ in $[0,1/8]$ and $0 \le \eta_1 \le 1$, and $\eta_2 \in C_c^\infty(B_1)$ be such that $\eta_2 \equiv 1$ in $B_{1/2}$ and $0 \le \eta_2 \le 1$. We define $\eta = \eta_1\eta_2$. Moreover, let $\bar\eta_1^{2s} \in C^{0,1}$ be such that $\bar\eta_1(0) = 0$, $\bar\eta_1 \equiv 1$ in $[1/2,1]$, and $0 \le \bar\eta_1 \le 1$. Let $\bar\eta_2 \in C_c^{1,1}(B_{1/4})$ be* *such that* $\bar\eta_2 \equiv 1$ *in* $B_{1/8}$ *and* $0 \le \bar\eta_2 \le 1$. *We define* $\bar\eta := \bar\eta_1\bar\eta_2$. *Then, there exists* $\sigma_0 = \sigma_0(n,s,\lambda,\Lambda,\|\bar\eta_1^{2s}\|_{C^{0,1}([0,1])},\|\bar\eta_2\|_{C^{1,1}(B_1)}) > 0$ *such that for every* $\sigma \ge \sigma_0$ *and* $u \in C^\infty((0,1)\times\mathbb{R}^n)$ *the following estimate holds true for* $\tilde u := \eta u$*:*

$$\begin{aligned}&(\partial_t + L_{K^{(t)}})(\bar\eta^2(\partial^2_{ee}\tilde u)^2_+ + \sigma(\partial_e\tilde u)^2)\\ &\qquad \le 2\bar\eta^2(\partial_t + L_{K^{(t)}})(\partial^2_{ee}\tilde u)(\partial^2_{ee}\tilde u)_+ + 2\sigma(\partial_t + L_{K^{(t)}})(\partial_e\tilde u)\partial_e\tilde u.\end{aligned} \tag{3.82}$$

Moreover, the following estimate holds true whenever all expressions are well-defined:

$$\begin{aligned}&(\partial_t + L_{K^{(t)}})(\bar\eta^2(D_{-h}D_h\tilde u)^2_+ + \sigma[(D_h\tilde u)^2]_{-h})\\ &\le 2\bar\eta^2(\partial_t + L_{K^{(t)}})(D_{-h}D_h\tilde u)(D_{-h}D_h\tilde u)_+ + 2\sigma[(\partial_t + L_{K^{(t)}})(D_h\tilde u)D_h\tilde u]_{-h}.\end{aligned} \tag{3.83}$$

Let us now explain how to apply the parabolic Bernstein key estimate Theorem 3.7.2 in order to obtain semiconvexity estimates for solutions to the parabolic nonlocal obstacle problem.

By combination of the parabolic key estimate Theorem 3.7.2 and the parabolic maximum principle, we obtain semiconvexity estimates in space for smooth solutions to the parabolic nonlocal obstacle problem:

Theorem 3.7.5 (semiconvexity estimate)**.** *Let* $s \in (0,1)$, $L_{K^{(t)}} \in \mathcal{L}_s(\lambda,\Lambda;2)$ *for any* $t \in (0,1)$. *Let* $f \in L^\infty((0,1);C^{1,1}(B_1))$. *Let* $u \in C^{1+s}_x((0,1)\times B_1) \cap C^{0,1}_t((0,1)\times B_1)$ *be a solution to the parabolic obstacle problem*

$$\min\{\partial_t u + L_{K^{(t)}}u - f, u\} = 0 \quad \textit{in } (0,1)\times B_1. \tag{3.84}$$

Then, for any $e \in \mathbb{S}^{n-1}$*, it holds*

$$\inf_{(1/2,1)\times B_{1/2}} \partial^2_{ee}u \ge -C\left(\|f\|_{L^\infty((0,1);C^{1,1}(B_1))} + \|u\|_{L^\infty((0,1)\times\mathbb{R}^n)}\right), \tag{3.85}$$

where $C = C(n,s,\lambda,\Lambda) > 0$ *is a constant.*

Proof. We split the proof into two parts.

Step 1: First order estimate:
Let η be as in Theorem 3.7.2. By the same arguments as in the proof of (3.54), we obtain in $(1/2,1)\times B_{1/2-|h|}$:

$$(\partial_t + L_{K^{(t)}})(D_hu)D_hu \le \|\nabla f\|_{L^\infty((0,1)\times B_1)}|D_hu|.$$

Moreover, since u solves (3.84), we have

$$[(\partial_t + L_{K^{(t)}})(u)u]_h \le \|f\|_{L^\infty((0,1)\times B_1)}\|u\|_{L^\infty((0,1)\times B_1)}.$$

Using (3.79), we obtain

$$\begin{aligned}(\partial_t + L_{K^{(t)}})(\eta^2(D_hu)^2 + \sigma[u^2]_h) &\le 2\eta^2(\partial_t + L_{K^{(t)}})(D_hu)D_hu + 2\sigma[(\partial_t + L_{K^{(t)}})(u)u]_h\\ &\le 2\eta^2\|\nabla f\|_{L^\infty((0,1)\times B_1)}|D_hu| + \|f\|_{L^\infty((0,1)\times B_1)}\|u\|_{L^\infty((0,1)\times B_1)}\end{aligned}$$

and deduce from the parabolic maximum principle:

$$\|D_hu\|^2_{L^\infty((1/2,1)\times B_{1/2-|h|})} \le C\left(\|f\|^2_{L^\infty((0,1);C^{0,1}(B_1))} + \|u\|^2_{L^\infty((0,1)\times\mathbb{R}^n)}\right). \tag{3.86}$$

Step 2: Second order estimate:
Let $\eta, \bar{\eta}$ be as in Corollary 3.7.4. By the same arguments as in the proof of (3.54) and (3.56), we obtain in $(1/2, 1) \times B_{1/4-|h|}$:

$$(\partial_t + L_{K^{(t)}})(D_h u) D_h u \le \|\nabla f\|_{L^\infty((0,1)\times B_1)} |D_h u|,$$
$$(\partial_t + L_{K^{(t)}})(D_{-h} D_h u)(D_{-h} D_h u)_+ \le \|D^2 f\|_{L^\infty((0,1)\times B_1)} (D_{-h} D_h u)_+.$$

Moreover, we have (see (3.57))

$$|(\partial_t + L_{K^{(t)}})(D_h[(1-\eta)u])| \le C\|u\|_{L^\infty((0,1)\times\mathbb{R}^n)},$$
$$|(\partial_t + L_{K^{(t)}})(D_{-h} D_h[(1-\eta)u])| \le C\|u\|_{L^\infty((0,1)\times\mathbb{R}^n)}.$$

Therefore, we obtain from (3.83)

$$\begin{aligned}
&(\partial_t + L_{K^{(t)}})(\bar{\eta}^2 (D_{-h} D_h \tilde{u})_+^2 + \sigma[(D_h \tilde{u})^2]_{-h}) \\
&\quad \le 2\bar{\eta}^2 (\partial_t + L_{K^{(t)}})(D_{-h} D_h \tilde{u})(D_{-h} D_h \tilde{u})_+ + 2\sigma[(\partial_t + L_{K^{(t)}})(D_h \tilde{u}) D_h \tilde{u}]_{-h} \\
&\quad \le C \left(\|f\|_{L^\infty((0,1);C^{1,1}(B_1))} + \|u\|_{L^\infty((0,1)\times\mathbb{R}^n)} \right) \left(\bar{\eta}^2 (D_{-h} D_h u)_+ + [|D_h \tilde{u}|]_{-h} \right),
\end{aligned}$$

and deduce from the parabolic maximum principle:

$$\begin{aligned}
\sup_{[1/2,1]\times B_{1/8}} (D_{-h} D_h u)_+^2 &\le \sup_{(1/2,1)\times B_{1/4-|h|}} [\bar{\eta}^2 (D_{-h} D_h \tilde{u})_+^2 + [(D_h \tilde{u})^2]_{-h}] \\
&\le C \left(\|f\|^2_{L^\infty((0,1);C^{1,1}(B_1))} + \|u\|^2_{L^\infty((0,1)\times\mathbb{R}^n)} + \|D_h u\|^2_{L^\infty([1/2,1)\times B_{1/4-|h|})} \right).
\end{aligned}$$

A combination of the previous estimate with the Lipschitz estimates (3.86) from Step 1 yields the desired semiconvexity estimate (3.85) upon taking the limit $h \to 0$. This proves the claim. □

3.7.2 Nonsymmetric operators and drift terms

The goal of this section is to explain how the Bernstein technique developed in this article can be extended to nonsymmetric nonlocal operators. Our study covers nonlocal operators with nonsymmetric jumping kernels that might possess a first order drift term.
Recall that so far, we have considered operators of the form

$$Lu(x) = \text{p.v.} \int_{\mathbb{R}^n} (u(x) - u(y)) K(x-y) \, dy,$$

where K was a symmetric jumping kernel, i.e., $K(y) = K(-y)$. From now on, we will drop this assumption. In order for the nonlocal operator to remain well-defined for smooth functions, if $s \ge 1/2$, we need to slightly adjust the definition of L, as follows:

$$Lu(x) = \begin{cases} \int_{\mathbb{R}^n} (u(x) - u(y)) K(x-y) \, dy, & \text{if } s \in (0, 1/2), \\ \text{p.v.} \int_{\mathbb{R}^n} (u(x) - u(y)) K(x-y) \, dy & \text{if } s = 1/2, \\ \int_{\mathbb{R}^n} (u(x) - u(y) - \nabla u(x) \cdot (x-y)) K(x-y) \, dy, & \text{if } s \in (1/2, 1). \end{cases} \tag{3.87}$$

For the p.v.-integral in case $s = 1/2$ to be well-defined, we need to add the following cancellation condition (see [76], [104]):

$$\int_{B_R \setminus B_r} yK(y)\,\mathrm{d}y = 0 \qquad \forall R > r > 0. \tag{3.88}$$

Moreover, we work under the assumption that K satisfies ($K_{\asymp}$) and (C^1) for some constants $0 < \lambda \le \Lambda$.
Let us mention the works [54], [128], where $C^{1,\varepsilon}$-regularity for solutions to fully nonlinear nonlocal equations governed by operators with nonsymmetric jumping kernels has been studied. Moreover, we refer to [76], where the interior and boundary regularity theory is developed for nonsymmetric stable operators.

A second way to introduce nonsymmetry to the picture is by considering operators with a drift, i.e., $L + b \cdot \nabla$, where $b \in \mathbb{R}^n$. The regularity theory for nonlocal operators with drifts has recently gained some attention, inspired by the works [53], [131], [132] on the critical dissipative SQG equation (see also [186]). The case $s \le 1/2$ is of particular interest, since the drift term becomes (super)critical with respect to the scaling of the equation. We refer the interested reader to [187], [188] for a detailed study of higher regularity properties under the presence of a critical or supercritical drift.

Under these conditions, we are able to establish Bernstein key estimates reminiscent of (3.8) and (3.10)

Theorem 3.7.6. *Let $s \in (0, 1)$, L be as in (3.87), K be as before, and $b \in \mathbb{R}^n$ with $|b| \le \Lambda$. Let $\eta \in C^{1,1}(\mathbb{R}^n)$ be such that $\eta^{2s} \in C^{0,1}(\mathbb{R}^n)$ and $\eta \ge 0$. Then, there exists $\sigma_0 = \sigma_0(n, s, \lambda, \Lambda, \|\eta\|_{C^{1,1}(\mathbb{R}^n)}, \|\eta^{2s}\|_{C^{0,1}(\mathbb{R}^n)}) > 0$ such that for every $\sigma \ge \sigma_0$ and $u, v \in C^{1+2s+\varepsilon}_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$:*

$$(L + b \cdot \nabla)(\eta^2(\partial_e u)^2 + \sigma u^2) \le 2\eta^2(L + b \cdot \nabla)(\partial_e u)\partial_e u + 2\sigma(L + b \cdot \nabla)(u)u, \tag{3.89}$$
$$(L + b \cdot \nabla)(\eta^2(\partial_e v)_+^2 + \sigma v^2) \le 2\eta^2(L + b \cdot \nabla)(\partial_e v)(\partial_e v)_+ + 2\sigma(L + b \cdot \nabla)(v)v. \tag{3.90}$$

Remark 3.7.7. (i) By following the same arguments as in Section 3.5, it is also possible to derive Bernstein key estimates in terms of difference quotients for $L + b \cdot \nabla$.

(ii) Recently, in [104], the regularity theory for the nonlocal obstacle problem has been extended to integro-differential operators of the form (3.87) with nonsymmetric kernels satisfying ($K_{\asymp}$). Moreover, we refer to [95], [104] ($s = 1/2$), [159], [111], [139] ($s > 1/2$), for regularity results on the nonlocal obstacle problem under the presence of a drift term. We expect it to be possible to obtain analogues to Theorem 3.1.3 and Theorem 3.1.4 also for such generalized problems by combining the ideas in the aforementioned papers with those in Section 3.6.

Before we provide a proof of Theorem 3.7.6, let us point out that the product rule Lemma 3.2.3 and the interpolation Lemma 3.3.4 remain valid for nonsymmetric kernels and that the bilinear form B keeps the same shape as in the symmetric case. Moreover, we have a nonsymmetric counterpart of Lemma 3.3.2, which will be applied exactly as before. However the proof of the first estimate changes slightly in case $s \le 1/2$. For completeness, we provide the results and a short proof below:

Lemma 3.7.8. *Let $s \in (0,1)$. Assume*

$$K(y) \le \Lambda |y|^{-n-2s}, \quad \mathrm{supp}(K) \subset B_\varepsilon$$

for some $\Lambda > 0$ and $\varepsilon \in (0,1)$. Let $\eta \in C^{1,1}(B_1)$. Then,

$$L(\eta^2)(x) \le \begin{cases} c_1 \|\eta^2\|_{C^{1,1}(B_\varepsilon(x))}(\varepsilon + \eta(x))\varepsilon^{1-2s}, & \text{if } s \le 1/2, \\ c_1 \|\eta^2\|_{C^{1,1}(B_\varepsilon(x))}\varepsilon^{2-2s}, & \text{if } s > 1/2, \end{cases}$$
$$B(\eta,\eta)(x) \le c_2 \|\nabla \eta\|^2_{L^\infty(B_\varepsilon(x))}\varepsilon^{2-2s},$$

where $c_1, c_2 > 0$ are constants depending only on n, s, Λ.

Proof. The proof of the second estimate remains exactly the same as in the symmetric case. Also the proof of the first estimate does not change if $s > 1/2$. To prove the first estimate in case $s \le 1/2$, observe that

$$\begin{aligned} L(\eta^2)(x) &= \int_{B_\varepsilon(x)} (\eta^2(x) - \eta^2(y) - \nabla(\eta^2)(x)\cdot(x-y))K(x-y)\,\mathrm{d}y \\ &\quad + \mathrm{p.v.}\int_{B_\varepsilon(x)} \nabla(\eta^2)(x)\cdot(x-y)K(x-y)\,\mathrm{d}y \\ &\le c_1 \|\eta^2\|_{C^{1,1}(B_\varepsilon(x))}\varepsilon^{2-2s} + |\nabla(\eta^2)(x)| \left| \mathrm{p.v.}\int_{B_\varepsilon} yK(y)\,\mathrm{d}y \right|. \end{aligned}$$

In case $s < 1/2$, we can estimate the second term using the pointwise upper bound for K by

$$|\nabla(\eta^2)(x)| \left| \mathrm{p.v.}\int_{B_\varepsilon} yK(y)\,\mathrm{d}y \right| \le c|\nabla(\eta^2)(x)|\varepsilon^{1-2s} \le c\|\eta\|_{C^{0,1}(B_\varepsilon(x))}\eta(x)\varepsilon^{1-2s}.$$

In case $s = 1/2$, the second term can be estimated by

$$|\nabla(\eta^2)(x)| \left| \mathrm{p.v.}\int_{B_\varepsilon} yK(y)\,\mathrm{d}y \right| \le c|\nabla(\eta^2)(x)| \le c\|\eta\|_{C^{0,1}(B_\varepsilon(x))}\eta(x)$$

due to the cancellation condition (3.88). This proves the desired result. □

We are now in the position to prove Theorem 3.7.6:

Proof of Theorem 3.7.6. Let us first explain how to prove (3.89). As in the proof of Theorem 3.1.6, for any x, we split the kernel $K = K_1 + K_2$ according to Lemma 3.3.1 with $\varepsilon = \gamma\eta(x)$, where $\gamma \in (0,1)$ is chosen as in the proof of Theorem 3.1.6. By carefully tracing the arguments in Step 1 and Step 2 of the proof of Theorem 3.1.6, it becomes apparent that Lemma 3.7.8 can be applied exactly in the same way as Lemma 3.3.2 and apart from that no further changes are necessary due to the lack of symmetry, so that we obtain

$$\int_{\mathbb{R}^n} (\eta^2(\cdot) - \eta^2(y))(\partial_e u(y))^2 K_2(\cdot - y)\,\mathrm{d}y \le \eta^2 \int_{\mathbb{R}^n} (\partial_e u(\cdot) - \partial_e u(y))^2 K_2(\cdot - y)\,\mathrm{d}y + \sigma_1 B_K(u,u)$$

and

$$L_{K_1}(\eta^2)(\partial_e u)^2 - B_{K_1}(\eta^2, (\partial_e u)^2) \le \frac{3}{4}\eta^2 B_{K_1}(\partial_e u, \partial_e u) + \sigma_2 B_K(u,u).$$

Let us explain how these two results imply the following estimate

$$L(\eta^2)(\partial_e u)^2 - B(\eta^2, (\partial_e u)^2) \le \frac{7}{8}\eta^2 B(\partial_e u, \partial_e u) + \sigma B(u,u), \tag{3.91}$$

which in turn implies the key estimate (3.89) with $b \equiv 0$, namely,

$$L(\eta^2(\partial_e u)^2 + \sigma u^2) \le 2\eta^2 L(\partial_e u)\partial_e u + 2\sigma L(u)u. \tag{3.92}$$

In case $s \le 1/2$, we are already done. In case $s > 1/2$, it remains to estimate

$$\begin{aligned}(\partial_e u(x))^2 \int_{\mathbb{R}^n} \nabla\eta^2(x)\cdot(x-y)K_2(x-y)\,\mathrm{d}y \\ \le \frac{1}{8}\eta^2(x) B_{K_1}(\partial_e u, \partial_e u)(x) + \sigma_3 B_K(u,u)(x),\end{aligned} \tag{3.93}$$

in order to obtain (3.91). The estimate (3.93) can be proved observing that by property (iv) of K_2 (see Lemma 3.3.1) and $(K_{\asymp})$:

$$\int_{B_1(x)} \nabla\eta^2(x)\cdot(x-y)K_2(x-y)\,\mathrm{d}y \le 2\|\eta\|_{C^{0,1}(\mathbb{R}^n)}\eta(x)\int_{B_1\setminus B_\varepsilon} |y|K(y)\,\mathrm{d}y \le C\eta(x)^{2-2s}.$$

Next, we apply the interpolation estimate Lemma 3.3.4 with $\delta = \left(\frac{1}{8C}\right)^{\frac{1}{2s}}\eta(x) \wedge \frac{\varepsilon}{2}$ and read off (3.93), as desired. We have therefore established (3.92) for all $s \in (0,1)$.

Let us finish the proof of (3.89) by explaining how to treat the drift term. To this end, let us compute

$$\begin{aligned}(L + b\nabla)(\eta^2(\partial_e u)^2 + \sigma u^2) &= 2(b\nabla\eta)\eta(\partial_e u)^2 \\ &\quad + 2\eta^2(L + b\nabla)(\partial_e u)(\partial_e u) + 2\sigma(L+b\nabla)(u)u \\ &\quad + L(\eta^2)(\partial_e u)^2 - B(\eta^2, (\partial_e u)^2) - \eta^2 B(\partial_e u, \partial_e u) - \sigma B(u,u).\end{aligned}$$

Therefore, we need to prove

$$2(b\nabla\eta)\eta(\partial_e u)^2 + L(\eta^2)(\partial_e u)^2 - B(\eta^2, (\partial_e u)^2) \le \eta^2 B(\partial_e u, \partial_e u) + \sigma B(u,u).$$

Since we already know (3.91) it remains to establish

$$2(b\nabla\eta)\eta(\partial_e u)^2 \le \frac{1}{8}\eta^2 B_{K_1}(\partial_e u, \partial_e u) + \sigma_4 B_{K_1}(u,u). \tag{3.94}$$

To do so, assume without loss of generality that $b\nabla\eta \neq 0$, otherwise the inequality is trivial. Let us apply the interpolation estimate Lemma 3.3.4 with $\delta = \left(\frac{\eta(x)}{16b\nabla\eta(x)}\right)^{\frac{1}{2s}} \wedge \frac{\varepsilon}{2}$ and proceed as in the proof of Theorem 3.7.2:

$$\begin{aligned}2(b\nabla\eta)\eta(\partial_e u)^2 &\le 2(b\nabla\eta)\eta\left(\delta^{2s}B_{K_1}(\partial_e u, \partial_e u) + c\delta^{2s-2}B_{K_1}(u,u)\right) \\ &\le \frac{1}{8}\eta^2 B_{K_1}(\partial_e u, \partial_e u) + c(b\nabla\eta)^{\frac{1}{s}}\eta^{2-\frac{1}{s}}B_{K_1}(u,u) \\ &\le \frac{1}{8}\eta^2 B_{K_1}(\partial_e u, \partial_e u) + c\eta_2^2 B_{K_1}(u,u).\end{aligned}$$

Since $(b\nabla\eta)^{\frac{1}{s}}\eta^{2-\frac{1}{s}} = c(s)(b\nabla(\eta^{2s}))^{\frac{1}{s}}$ is bounded, this yields (3.94), as desired. This concludes the proof of the Bernstein key estimate (3.89).

Finally, in order to prove (3.90), one makes the same adaptation to the aforementioned proof as in the derivation of Theorem 3.1.8. Note that the existing of a drift term does not require any changes. However, in order to deal with the lack of symmetry of K, it is worth pointing out that in order to adapt Lemma 3.3.8 to the nonsymmetric case for $s > 1/2$, one has to use that by the regularity of v it holds $\nabla(\partial_e v(x))_- = 0$, whenever $\partial_e v(x) > 0$, in order to obtain

$$\begin{aligned}
&\int_{B_\delta(x)} (\partial_e v(y))_- K(x-y)\,\mathrm{d}y \\
&\quad = -\int_{B_\delta(x)} \Big((\partial_e v(x))_- - (\partial_e v(y))_- - \nabla(\partial_e v(x))_- \cdot (x-y)\Big) K(x-y)\,\mathrm{d}y \\
&\quad \le -L((\partial_e v)_-)(x),
\end{aligned}$$

which complements the computation in (3.25). □

3.7.3 General Lévy kernels

The goal of this section is to extend the Bernstein technique to nonlocal operators of the form

$$Lu(x) = \text{p.v.} \int_{\mathbb{R}^n} (u(x) - u(y))K(x-y)\,\mathrm{d}y,$$

where K is a kernel that satisfies a comparability condition with respect to a general radial Lévy kernel but is not necessarily comparable to the fractional Laplacian. Instead of ($K_\asymp$), we assume that

$$\lambda |y|^{-n} g(|y|) \le K(y) \le \Lambda |y|^{-n} g(|y|) \qquad \forall y \in \mathbb{R}^n \tag{3.95}$$

for some strictly decreasing function $g : (0,\infty) \to [0,\infty)$ satisfying

$$2s_1 r^{-1} g(r) \le |g'(r)| \le 2s_2 r^{-1} g(r) \qquad \forall r > 0 \tag{3.96}$$

for some $0 < s_1 \le s_2 < 1$. Note that (3.96) implies the following doubling properties for g:

$$\begin{aligned}
g(\lambda r) &\le \lambda^{-2s_2} g(r) \qquad \forall \lambda < 1, \\
g(\lambda r) &\le \lambda^{-2s_1} g(r) \qquad \forall \lambda > 1.
\end{aligned}$$

In particular, we have

$$\int_{\mathbb{R}^n} \min\{1, |y|^2\} |y|^{-n} g(y)\,\mathrm{d}y < \infty, \qquad \lim_{r \searrow 0} g(r) = \infty, \tag{3.97}$$

which implies that K is a Lévy kernel.

The study of nonlocal operators L satisfying (3.97) whose jumping kernel K is not comparable to the one of the fractional Laplacian are of general interest since they arise as generators of Lévy processes and in particular of subordinate Brownian motion. In order to study properties of harmonic functions with respect to L, it is natural to impose some growth or scaling conditions such as (3.96) on the kernel, or on the Fourier symbol of L. Let us mention for instance the works

[141, 113], where gradient estimates for L-harmonic functions and [59, 60, 28, 129, 115, 114], where estimates for fundamental solutions of the associated Cauchy problem are derived, using a probabilistic approach.

We claim that the key estimates (3.8) and (3.10) remain true in this general setting:

Theorem 3.7.9. *Let $0 < s_1 \le s_2 < 1$ and g be such that (3.96) holds true and assume that K satisfies (3.95) and (C^1). Let $\eta \in C^{1,1}(\mathbb{R}^n)$ be such that $\eta \ge 0$. Then, there exists $\sigma_0 = \sigma_0(n, s_1, s_2, \Lambda/\lambda, \|\eta\|_{C^{1,1}(\mathbb{R}^n)}) > 0$ such that for every $\sigma \ge \sigma_0$ and every smooth enough $u, v \in L^\infty(\mathbb{R}^n)$*

$$L\Big(\eta^2(\partial_e u)^2 + \sigma u^2\Big) \le 2\eta^2 L(\partial_e u)\partial_e u + 2\sigma L(u)u \qquad \text{in} \quad \mathbb{R}^n, \tag{3.98}$$

$$L\Big(\eta^2(\partial_e v)_+^2 + \sigma v^2\Big) \le 2\eta^2 L(\partial_e v)(\partial_e v)_+ + 2\sigma L(v)v \qquad \text{in} \quad \mathbb{R}^n. \tag{3.99}$$

In our setting, the statement of the interpolation estimate reads as follows:

Lemma 3.7.10. *Let $0 < s_1 \le s_2 < 1$ and $\delta \in (0,1)$. Assume that K satisfies for some $0 < \lambda \le \Lambda$:*

$$\lambda|y|^{-n}g(|y|) \le K(y) \le \Lambda|y|^{-n}g(|y|) \quad \forall y \in B_\delta, \tag{3.100}$$

$$|\nabla K(y)| \le \Lambda|y|^{-1}K(y) \quad \forall y \in B_\delta, \tag{3.101}$$

where g satisfies (3.96). Then, for every $x \in \mathbb{R}^n$ and $u \in C^{0,1}(B_\delta(x))$ it holds

$$\Big(\partial_e u(x)\Big)^2 \le g(\delta)^{-1}B_K(\partial_e u, \partial_e u)(x) + c\delta^{-2}g(\delta)^{-1}B_K(u,u)(x),$$

$$(\partial_e v(x))_+^2 \le \Big[g(\delta)^{-1}B((\partial_e v)_+, (\partial_e v)_+)(x) - L((\partial_e v)_-)(x)(\partial_e v)_+(x)\Big] + c\delta^{-2}g(\delta)^{-1}B(v,v)(x),$$

where $c = c(n, s_1, s_2, \lambda, \Lambda) > 0$ does not depend on δ.

Proof. We only explain how to construct K_δ. Then, the proof of the estimates goes in the same way as the proof of Lemma 3.3.4 and Lemma 3.3.8, replacing δ^{2s} by $g(\delta)^{-1}$. We define

$$K_\delta(y) = \psi(|y|/\delta)K(y)|y|^{\frac{n}{2}+1}g(|y|)^{-\frac{1}{2}}.$$

Note that the properties (1), (2), and (3) from the proof of Lemma 3.3.4 remain true for this choice of K_δ. To see (2), we compute using (3.96), (3.100), and (3.101):

$$\begin{aligned}|\nabla K_\delta(y)|^2 &\lesssim (|y|/\delta)^2|\psi'(|y|/\delta)|K^2(y)|y|^n g(|y|)^{-1} \\ &\quad + \psi^2(|y|/\delta)\Big[|\nabla K(y)|^2|y|^{n+2}g(|y|)^{-1} + K^2(y)|y|^n g(|y|)^{-1} + K^2(y)|y|^{n+2}|g'(|y|)|^2 g(|y|)^{-3}\Big] \\ &\lesssim K(y).\end{aligned}$$

Let us make the following observation, which follows from (3.96) (resp. its doubling properties):

$$\int_{B_{2r}\setminus B_r} K(y)\,\mathrm{d}y \asymp \int_r^{2r} t^{-1}g(t)\,\mathrm{d}t \asymp -\int_r^{2r} g'(t)\,\mathrm{d}t \asymp g(r) - g(2r) \asymp g(r), \quad \forall r > 0, \tag{3.102}$$

Therefore, using again the doubling properties of g, as well as that $0 < s_1, s_2 < 1$, we have

$$\begin{aligned}\mu_{K_\delta}(B_\delta) &= \sum_{k=0}^\infty \int_{B_{\delta 2^{-k}}\setminus B_{\delta 2^{-k-1}}} K_\delta(y)\,\mathrm{d}y \asymp \sum_{k=0}^\infty (\delta 2^{-k})^{\frac{n}{2}+1} g(\delta 2^{-k})^{-1/2} \int_{B_{\delta 2^{-k}}\setminus B_{\delta 2^{-k-1}}} K(y)\,\mathrm{d}y \\ &\asymp \sum_{k=0}^\infty (\delta 2^{-k})^{\frac{n}{2}+1} g(\delta 2^{-k})^{1/2} \asymp \delta^{\frac{n}{2}+1}g(\delta)^{1/2}.\end{aligned}$$

The latter estimate can be seen as a counterpart of property (4) in the proof of Lemma 3.3.4. □

Moreover, we have the following replacement of Lemma 3.3.2:

Lemma 3.7.11. *Let* $0 < s_1 \le s_2 < 1$ *and* K *be symmetric, with*

$$K(y) \le \Lambda |y|^{-n} g(y), \quad \mathrm{supp}(K) \subset B_\varepsilon \tag{3.103}$$

for some $\Lambda > 0$ *and* $\varepsilon \in (0,1)$, *where* g *satisfies* (3.96). *Let* $\eta \in C^{1,1}(B_1)$. *Then, for any* $x \in B_1$

$$\begin{aligned} L(\eta^2)(x) &\le c_1 \|D^2\eta^2\|_{L^\infty(B_\varepsilon(x))} \varepsilon^2 g(\varepsilon), \\ B(\eta,\eta)(x) &\le c_2 \|\nabla \eta\|^2_{L^\infty(B_\varepsilon(x))} \varepsilon^2 g(\varepsilon), \end{aligned}$$

where $c_1, c_2 > 0$ *are constants depending only on* n, s_1, s_2, Λ.

Proof. The proof follows along the lines of the proof of Lemma 3.3.2, using (3.103) and the doubling properties of g to estimate:

$$\begin{aligned} \int_{B_\varepsilon} |y|^2 K(y) \,\mathrm{d}y &\lesssim \sum_{k=0}^{\infty} \int_{B_{\varepsilon 2^{-k}} \setminus B_{\varepsilon 2^{-k-1}}} |y|^{2-n} g(|y|) \,\mathrm{d}y \\ &\lesssim \sum_{k=0}^{\infty} (\varepsilon 2^{-k})^2 g(\varepsilon 2^{-k}) \lesssim \varepsilon^2 g(\varepsilon) \sum_{k=0}^{\infty} (2^{-k})^{2-2s_2} \lesssim \varepsilon^2 g(\varepsilon). \end{aligned}$$

□

Finally, we can give the:

Proof of Theorem 3.7.9. The proof goes in the exact same way as the proofs of Theorem 3.1.6 and Theorem 3.1.8. We only need to replace the interpolation estimates Lemma 3.3.4 and Lemma 3.3.8 by Lemma 3.7.10, and the cut-off estimate Lemma 3.3.2 by Lemma 3.7.11. Moreover, note that Lemma 3.3.1 remains true in this generalized setup. We apply the interpolation estimate and kernel decomposition with a suitable choice of δ and ε, which requires g to be invertible. Note that it is possible to invert g since it is strictly decreasing and by (3.97), we have that $g(0) = \infty$ and $g(\infty) = 0$. □

3.8 Appendix

The goal of this section is to give the proof of Lemma 3.6.5. Our proof relies on a modification of the ideas from [1] and [77].
The main auxiliary result in the proof of Lemma 3.6.5 is the following variant of Lemma 3.6 from [1] for the nonlocal obstacle problem:

Lemma 3.8.1. *Let* $s \in (0,1)$, $L \in \mathcal{L}_s(\lambda, \Lambda; 1)$, *and* $\alpha \in (0,s)$. *Assume that* u *with* $u \not\equiv 0$ *in* $B_{1/2}$ *is a solution to*

$$\min\{Lu - f, u\} = 0 \quad \text{in } B_1,$$

where $f \in C^{\beta-2s}(B_1)$ *for some* $\beta \in (2s, 1+s)$. *Then,* $\tilde{u} = u \mathbb{1}_{B_2}$ *solves the obstacle problem*

$$\min\{L\tilde{u} - \tilde{f}, \tilde{u}\} = 0 \quad \text{in } B_1,$$

for some $\tilde{f} \in C^{\beta-2s}(B_1)$ *with*

$$\|\tilde{f}\|_{C^{\beta-2s}(B_1)} \le C \left([f]_{C^{\beta-2s}(B_1)} + \left\| \frac{u}{(1+|\cdot|)^{1+s+\alpha}} \right\|_{L^\infty(\mathbb{R}^n)} \right), \tag{3.104}$$

where $C = C(n, s, \lambda, \Lambda) > 0$ *is a constant.*

Proof of Lemma 3.8.1. We follow the proof of Lemma 3.6 in [1]:
First, we observe that $\tilde{u}$ satisfies by assumption:

$$\begin{cases} L\tilde{u} &= \tilde{f} \quad \text{in } B_1 \cap \{\tilde{u} > 0\}, \\ L\tilde{u} &\ge \tilde{f} \quad \text{in } B_1 \cap \{\tilde{u} = 0\}. \end{cases}$$

where

$$\tilde{f} = -L(u\mathbb{1}_{\mathbb{R}^n \setminus B_2}) + f.$$

By following the same arguments as in the proof of Lemma 3.6 in [1], and observing that we can add and subtract constants to $\tilde{f}$ without affecting the left hand side of the following estimate, we obtain using (C^1):

$$\||h|D_h\tilde{f}\|_{L^\infty(B_{1-|h|})} \le C \left(\operatorname*{osc}_{B_1} f + |h| \left\| \frac{u}{(1+|\cdot|)^{1+s+\alpha}} \right\|_{L^\infty(\mathbb{R}^n)} \right).$$

Therefore, we can write

$$\tilde{f} = g + p,$$

where $p \in \mathbb{R}$ and $g \in L^\infty(B_1)$ satisfies

$$\|g\|_{L^\infty(B_1)} \le C \left(\operatorname*{osc}_{B_1} f + \left\| \frac{u}{(1+|\cdot|)^{1+s+\alpha}} \right\|_{L^\infty(\mathbb{R}^n)} \right). \tag{3.105}$$

Note that by Lemma 3.8.2, $|p|$ satisfies the same upper estimate as $\|g\|_{L^\infty(B_1)}$. Moreover, note that since $f \in C^{\beta-2s}(B_1)$, we can use analogous arguments as in the proof of (3.105) to deduce

$$\|g\|_{C^{\beta-2s}(B_1)} \le C \left([f]_{C^{\beta-2s}(B_1)} + \left\| \frac{u}{(1+|\cdot|)^{1+s+\alpha}} \right\|_{L^\infty(\mathbb{R}^n)} \right).$$

Altogether, we obtain the desired result. □

Lemma 3.8.2. *Let* $s \in (0,1)$, $L \in \mathcal{L}_s(\lambda, \Lambda)$. *Assume that* u *with* $u \neq 0$ *in* $B_{1/2}$ *is a solution to*

$$\begin{aligned} \min\{Lu - g - p, u\} = 0 \quad & \text{in } B_1, \\ u \equiv 0 \quad & \text{in } \mathbb{R}^n \setminus B_2, \end{aligned}$$

where $g \in C^{\beta-2s}(B_1)$ *for some* $\beta \in (2s, 1+s)$ *and* $p \in \mathbb{R}$. *Then, it holds*

$$|p| \le C \left(\|g\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)} \right), \tag{3.106}$$

where $C = C(n, s, \lambda, \Lambda) > 0$ *is a constant.*

Proof. Assume that (3.106) does not hold true. Then there exist sequences $(L_k) \subset \mathcal{L}_s(\lambda, \Lambda)$, $(g_k) \subset L^\infty(B_1)$, $(p_k) \subset \mathbb{R}$, and $(u_k) \subset L^\infty(\mathbb{R}^n)$ with $u_k \not\equiv 0$ in $B_{1/2}$, and $u_k \equiv 0$ in $\mathbb{R}^n \setminus B_2$, such that

$$\begin{aligned} \min\{L_k u_k - g_k - p_k, u_k\} &= 0 \quad \text{in } B_1, \\ \|u_k\|_{L^\infty(\mathbb{R}^n)} &\to 0, \\ \|g_k\|_{L^\infty(B_1)} &\to 0, \\ |p_k| &\to 1. \end{aligned} \tag{3.107}$$

We can extract subsequences such that $L_{k_m} \to L$ weakly (using a similar argument as in [166, Lemma 3.1]), $p_{k_m} \to p$ with $|p| = 1$, $g_{k_m} \to g = 0$ in $L^\infty(B_1)$, $u_{k_m} \to u = 0$ in $L^\infty(\mathbb{R}^n)$. Moreover, since by [39, Theorem 5.1] we get that $\|u_k\|_{C^{2s+\varepsilon}(B_{1/2})} \le C$, it holds $u_k \to u$ in $C^{2s+\varepsilon}(B_{1/2})$ (up to a subsequence) by Arzelà-Ascoli, and therefore $|L_k u_k| = |L_k u_k - Lu| \to 0$ locally uniformly in B_1.
Consequently,

$$\min\{Lu - g - p, u\} = 0 \quad \text{in } B_1.$$

In particular, $\min\{-p, 0\} = 0$ in B_1. This is a contradiction if $p = 1 > 0$. If $p = -1 < 0$, then, there must be $k \in \mathbb{N}$ such that $p_k < -\frac{3}{4}$, $\|g_k\|_{L^\infty(B_1)} < \frac{1}{8}$, and $\|L_k u_k\|_{L^\infty(B_{1/2})} < \frac{1}{8}$. However, since $u_k \not\equiv 0$ in $B_{1/2}$ by assumption, this contradicts (3.107). Therefore, (3.106) holds true. □

We are now in the position to give the proof of Lemma 3.6.5.

Proof of Lemma 3.6.5. Note that we can assume $u \not\equiv 0$ in $B_{1/2}$ without loss of generality, since otherwise there is nothing to prove. We define $\tilde{u} = u \mathbb{1}_{B_2}$ and deduce from Lemma 3.8.1 that $\tilde{u}$ solves

$$\min\{L\tilde{u} - \tilde{f}, \tilde{u}\} = 0 \quad \text{in } B_1.$$

By application of Theorem 5.1 in [39] together with (3.104) we obtain

$$\begin{aligned} \|u\|_{C^{\max\{2s+\varepsilon, 1+\varepsilon\}}(B_{1/2})} &= \|\tilde{u}\|_{C^{\max\{2s+\varepsilon, 1+\varepsilon\}}(B_{1/2})} \\ &\le C \left(\|\tilde{f}\|_{C^{\beta-2s}(B_1)} + \left\| \frac{\tilde{u}}{(1 + |\cdot|)^{n+2s}} \right\|_{L^1(\mathbb{R}^n)} \right) \\ &\le C \left([f]_{C^{\beta-2s}(B_1)} + \left\| \frac{u}{(1 + |\cdot|)^{1+s+\alpha}} \right\|_{L^\infty(\mathbb{R}^n)} \right). \end{aligned}$$

This proves the desired result. □

Part II

Boundary Harnack inequalities with right-hand side

I still don't understand the maths.
It's going to take me ten thousand years to understand it.[5]

[5] From Tamsyn Muir, *Nona the Ninth* (2022).

Introduction to Part II

The boundary Harnack is a key estimate that relates the growth of harmonic functions near zero Dirichlet boundary conditions with the geometry of the domain, in the following sense:

Theorem (Boundary Harnack)**.** *Let u and v be harmonic and positive in a regular enough domain Ω, with $0 \in \partial\Omega$, and assume that they vanish continuously on $\partial\Omega \cap B_1$. Let $p \in \Omega \cap B_1$, and assume that $u(p) = v(p)$.*

Then,

$$\frac{1}{C} \le \frac{u}{v} \le C \quad \text{in } \Omega \cap B_{1/2}.$$

Moreover,

$$\left\| \frac{u}{v} \right\|_{C^{0.\alpha}(\Omega \cap B_{1/2})} \le C.$$

Remark. The name Boundary Harnack refers in fact to two estimates of u/v, one in L^∞ and one in $C^{0.\alpha}$. We will refer to them as L^∞ boundary Harnack and $C^{0.\alpha}$ boundary Harnack throughout this Introduction to Part II. In the literature, the L^∞ boundary Harnack is sometimes called Carleson estimate, which is another very related result. The origin of the confusion is that, in some usual settings, the Carleson estimate implies the L^∞ boundary Harnack, which in turn implies the $C^{0.\alpha}$ boundary Harnack in a standard way. However, for parabolic equations the L^∞ boundary Harnack does not imply the $C^{0.\alpha}$ boundary Harnack (see [75]).

In a sense, this result is saying that no matter how different u and v are, they are converging to zero at the same rate. This can be seen as a natural generalization of the Hopf-Oleinik Lemma, that says that when the boundary of the domain is sufficiently smooth, positive solutions separate *linearly* from it.

Theorem (Hopf-Oleinik Lemma)**.** *Let Ω be a smooth domain, with $0 \in \partial\Omega$, and let u be a positive harmonic function in Ω, vanishing on $\partial\Omega \cap B_1$. Then,*

$$\frac{\partial u}{\partial \nu} > 0,$$

where ν is the unit inner normal vector to $\partial\Omega$.

If, instead, we consider a Lipschitz domain, we obtain that harmonic functions grow as a power of the distance to the zero Dirichlet boundary conditions, as it can be seen in the following example:

Example. Let $\Omega = (0,1)^2$. Then, $u = xy$ is harmonic in Ω, vanishing in $\partial\Omega \cap B_1$, and $u(x,x) = x^2$, that is, u grows quadratically as we approach the origin.

More generally, let $\Omega \subset \mathbb{R}^n$ be a Lipschitz cone with vertex at the origin, and let $\Gamma = \Omega \cap \mathbb{S}^{n-1}$. Then, we can look for a positive homogeneous harmonic function in this cone as follows:

Let us consider the ansatz $u(r, \theta) = r^\beta \varphi(\theta)$ in spherical coordinates. Then, the fact that u is harmonic becomes

$$u_{rr} + \frac{n-1}{r} u_r + \frac{1}{r^2} \Delta_{\mathbb{S}^{n-1}} u = 0 \quad \text{in } \Omega,$$

and therefore

$$\left(\beta(\beta - 1) + (n-1)\beta + \Delta_{\mathbb{S}^{n-1}}\right) \varphi = 0 \quad \text{on } \Gamma,$$

and then $\beta(\beta + n - 2) = \lambda$, where $\lambda \geq 0$ is the first eigenvalue of the Laplace-Beltrami operator on Γ. From here, getting the unique positive solution for β gives a unique (up to a multiplicative constant) positive homogeneous harmonic function in a cone, and the degree of homogeneity depends only on the geometry.

After seeing these examples, one would be tempted to write something like the following:

Not a theorem. *Let Ω be a Lipschitz domain with $0 \in \partial\Omega$, and let u be a positive harmonic function in Ω, vanishing on $\partial\Omega \cap B_1$. Then, there exists $\beta \geq 0$ such that for every $e \in \mathbb{S}^{n-1}$,*

$$\hat{u}(e) = \lim_{r \to 0} \frac{u(re)}{r^\beta}$$

is positive and finite.

In the case of smooth domains, we would have $\beta = 1$, and then $\hat{u}$ would coincide with the directional derivatives at the boundary.

So far so good. The problem with such a result is that it is false. Indeed, the existence of such β suggests that the domain *looks like a cone with the same aperture at every sufficiently small scale*. But it is not hard to think of (fractal) Lipschitz domains that look like cones with different apertures as $r \to 0$.

Instead of comparing one solution to a power of the distance, what we can actually do is say that *all* solutions have the same rate of growth near zero boundary conditions, and we can do so in a quantitative sense. Furthermore, we see that the quotient of positive harmonic functions is not only bounded, but Hölder continuous. In a sense, this means that it is enough to understand one solution to understand all solutions near the boundary.

The boundary Harnack in Lipschitz domains

There are many versions of the boundary Harnack, in different classes of domains, and with different assumptions on the operators. Here, we present an elliptic and a parabolic version in Lipschitz domains.

Let $\mathcal{L}$ be a uniformly elliptic operator, either in divergence or non-divergence form, that is,

$$\mathcal{L}u = \mathrm{Div}(A(x)\nabla u), \text{ or } \mathcal{L}u = \mathrm{Tr}(A(x)D^2 u),$$

where $\lambda I \leq A(x) \leq \Lambda I$ for some $0 < \lambda \leq \Lambda < +\infty$.

We will write $x = (x', x_n) \in \mathbb{R}^n$, and B_1' will be the unit ball of $\mathbb{R}^{n-1}$. Then, given a Lipschitz function $g : B_1' \to \mathbb{R}$ with $g(0) = 0$, we define for $r \in (0, 1]$

$$C_r := \{(x', x_n) : x' \in B_r',\ g(x') < x_n < g(x') + r\}.$$

In this formulation, the L^∞ boundary Harnack is the following.

Theorem ([73, Theorem 1.1]). *Let u and v be positive solutions to $\mathcal{L}u = \mathcal{L}v = 0$ in C_1, vanishing continuously on $\{x_n = g(x')\}$. Assume that $u(e_n/2) = v(e_n/2) = 1$. Then,*

$$C^{-1} \le \frac{u}{v} \le C \quad \text{in } C_{1/2},$$

with C depending only on the dimension, λ, Λ, and $\|g\|_{C^{0,1}}$.

In the elliptic setting, the Hölder continuity of the quotient follows by iterating the previous result, see [97, Appendix B].

Corollary. *Let u and v be positive solutions to $\mathcal{L}u = \mathcal{L}v = 0$ in C_1, vanishing continuously on $\{x_n = g(x')\}$. Assume that $u(e_n/2) = v(e_n/2) = 1$. Then,*

$$\left\| \frac{u}{v} \right\|_{C^{0,\alpha}(C_{1/2})} \le C,$$

where α and C are positive, and depend only on the dimension, λ, Λ, and $\|g\|_{C^{0,1}}$.

For parabolic equations, we consider domains as follows. Let $\tilde{g} : B_1' \times (-1,1) \to \mathbb{R}$ with $\tilde{g}(0,0) = 0$. Then, we define for $r \in (0,1]$

$$\tilde{C}_r := \{(x', x_n, t) : x' \in B_r',\ t \in (-r^2, r^2),\ \tilde{g}(x',t) < x_n < \tilde{g}(x',t) + r\}.$$

Here we assume that $\tilde{g} \in C^{1,\frac{1}{2}}_{x,t}$, that is,

$$\|\tilde{g}\|_{C^{1,\frac{1}{2}}_{x,t}} := \|\tilde{g}\|_{L^\infty} + \sup_{(x,t)\neq(y,s)} \frac{|\tilde{g}(x,t) - \tilde{g}(y,s)|}{|x-y| + |t-s|^{1/2}} < +\infty.$$

Notice that this norm corresponds to a Lipschitz norm for the parabolic distance, so it induces a natural notion of parabolic Lipschitz domain.

We also define the point in the past and the point in the future as

$$\underline{E} := \left(0, \frac{1}{2} + \tilde{g}\left(0, -\frac{1}{2}\right), -\frac{1}{2}\right), \text{ and } \overline{E} := \left(0, \frac{1}{2} + \tilde{g}\left(0, \frac{1}{2}\right), \frac{1}{2}\right).$$

Then, we have the parabolic L^∞ boundary Harnack.

Theorem ([75, Theorem 1.1]). *Let u and v be positive solutions to $u_t - \mathcal{L}u = v_t - \mathcal{L}v = 0$, vanishing continuously on $\tilde{C}_1 \cap \{x_n = \tilde{g}(x',t)\}$. Then,*

$$\frac{u}{v} \le C \frac{u(\overline{E})}{v(\underline{E})} \quad \text{in } \tilde{C}_{1/2},$$

with C depending only on the dimension, λ, Λ, and $\|\tilde{g}\|_{C^{1,\frac{1}{2}}_{x,t}}$.

Applications

Our motivation to study boundary Harnack inequalities stems from their usefulness in proving regularity of free boundaries in free boundary problems. The boundary Harnack also has important consequences in potential theory and probability.

Free boundary problems

The obstacle problem

The use of the boundary Harnack inequality in free boundary problems follows from an original idea of Athanasopoulos and Caffarelli. In [6], they used it to deduce that sufficiently flat Lipschitz free boundaries are actually $C^{1,\alpha}$ in the obstacle problem.

The technique works as follows. Consider a solution to the obstacle problem

$$\begin{cases} \Delta u &= f\chi_{\{u>0\}} \\ u &\geq 0, \end{cases}$$

where $\{u = 0\}$ is the contact set and $\partial\{u = 0\}$ is the free boundary, which we assume to be a Lipschitz graph around the origin. Recall that $u \in C^{1,1}$.

Then, let ν be the unit normal vector to the level sets $\{u = t\}$ for $t > 0$. Hence, we can write

$$\nu = \frac{\nabla u}{|\nabla u|} = \frac{\big(u_1/u_n, \ldots, u_{n-1}/u_n, 1\big)}{\sqrt{1 + \sum_{i=1}^{n-1} (u_i/u_n)^2}},$$

where we have chosen the orientation that makes ν point towards u increasing.

Now, if we prove that $u_i/u_n \in C^{0,\alpha}$, we will have that $\nu \in C^{0,\alpha}(\{u > 0\})$, and then we will be able to extend it to the free boundary continuously. Moreover, we deduce that the free boundary is $C^{1,\alpha}$.

On the other hand, the derivatives of u satisfy

$$\begin{cases} \Delta u_i &= f_i \quad \text{in} \quad \{u > 0\} \\ u_i &= 0 \quad \text{on} \quad \partial\{u > 0\}, \end{cases}$$

Then, with this argument, we can reduce the problem of understanding the regularity of the free boundary to understanding the regularity of the quotients of solutions to the Poisson equation. This we can do using the boundary Harnack. Indeed, $u_i/u_n \in C^{0,\alpha}$ for every coordinate i, and then $\partial\{u > 0\}$ is $C^{1,\alpha}$.

In the original applications, this approach was only available when f was constant, because the classical boundary Harnack concerns harmonic functions, or, more in general, solutions to elliptic equations with zero right-hand side. Extending this argument to non-constant f has been one of the main motivations in the development of boundary Harnack inequalities for equations with right-hand side [3, 2].

Higher regularity of free boundaries

There is a higher order analogue of the boundary Harnack that can also be used to prove higher order regularity of free boundaries. In summary, the quotient of two harmonic functions vanishing on the boundary of a $C^{k,\alpha}$ domain is $C^{k,\alpha}$. For simplicity, we will use the same definition as before for Lipschitz domains, but now with $g \in C^{k,\alpha}$.

Theorem (Higher order boundary Harnack, [71, Theorem 1.1])**.** *Let Ω be a $C^{k,\alpha}$ domain, and assume that $0 \in \partial\Omega$. Let $u > 0$ and v be two harmonic functions in $\Omega \cap B_1$ that vanish* *continuously on $\partial\Omega \cap B_1$. Assume that u is normalized so that $u(e_n/2) = 1$. Then,*

$$\left\| \frac{v}{u} \right\|_{C^{k,\alpha}(\Omega\cap B_{1/2})} \leq C \|v\|_{L^\infty(\Omega\cap B_1)},$$

where C depends only on k, α and the dimension.

Note that, by the known boundary Schauder estimates, both u and v are of class $C^{k,\alpha}$ up to the boundary. Moreover, since $u = v = 0$ on the boundary, and $\partial_n u > 0$ in a neighbourhood of $\partial\Omega$ by the Hopf-Oleinik lemma and continuity, we would deduce that the quotient belongs to $C^{k-1,\alpha}$.

The extra derivative that we gain with this result allows us to bootstrap the regularity of the free boundary up to C^∞ in the obstacle problem. Let $f \equiv 1$ for simplicity, and consider a solution to

$$\begin{cases} \Delta u &= \chi_{\{u>0\}} \\ u &\geq 0. \end{cases}$$

Assume that the free boundary is a $C^{1,\alpha}$ graph around the origin. Then, the derivatives of u are harmonic in $\{u > 0\}$, by the higher order boundary Harnack we have that $u_i/u_n \in C^{1,\alpha}$, and hence the normal vector

$$\nu = \frac{\big(u_1/u_n, \ldots, u_{n-1}/u_n, 1\big)}{\sqrt{1 + \sum\limits_{i=1}^{n-1} (u_i/u_n)^2}}$$

also belongs to $C^{1,\alpha}$, which implies that the free boundary is actually $C^{2,\alpha}$. Now we can repeat this procedure, and with each iteration we gain one derivative, so we conclude that the free boundary is C^∞.

Potential theory

In this section we will examine the connection of the boundary Harnack with potential theory. We start by introducing the notions of Green function and harmonic measure. We refer the interested reader to the lecture notes [163] for an introduction to the topic.

Definition. Given an open Lipschitz bounded domain Ω, and $x \in \Omega$, the Green function with pole at x is the solution to

$$\begin{cases} \Delta g &= -\delta_x \quad \text{in} \quad \Omega \\ g &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where δ_x is the Dirac delta, and we define the harmonic measure with pole at x as the unique Radon measure ω^x on $\partial\Omega$ such that

$$u(x) = \int_{\partial\Omega} f \mathrm{d}\omega^x,$$

where u is the solution to the Dirichlet problem

$$\begin{cases} \Delta u &= 0 \quad \text{in} \quad \Omega \\ u &= f \quad \text{on} \quad \partial\Omega. \end{cases}$$

We intentionally keep the notion of solution imprecise so to explain the ideas without introducing too much theory.

A fundamental result in the understanding of harmonic measure is the following theorem of Dahlberg [68].

Theorem. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ with Lipschitz constant L, and let σ be the surface measure on $\partial\Omega$. Assume that the origin belongs to $\partial\Omega$. Let $r > 0$, and $x_0 \in \Omega$ such that* $\mathrm{dist}(x_0, B_{2r} \cap \partial\Omega) \geq cr > 0$. *Then, the following holds:*

- *The harmonic measure ω^{x_0} and σ are mutually absolutely continuous.*
- *We have*
$$\left(\fint_{\partial\Omega\cap B_r} \left(\frac{\mathrm{d}\omega^{x_0}}{\mathrm{d}\sigma}\right)^2 \mathrm{d}\sigma\right)^{1/2} \leq C \fint_{\partial\Omega\cap B_r} \frac{\mathrm{d}\omega^{x_0}}{\mathrm{d}\sigma}\mathrm{d}\sigma = C\frac{\omega^{x_0}(B_r)}{\sigma(B_r)}.$$

The constant C is positive and depends only on L, c, and the dimension.

In the case of smooth domains,
$$\frac{\mathrm{d}\omega^{x_0}}{\mathrm{d}\sigma} = -\partial_\nu g,$$
where g is the Green function with pole at x_0, and the fact that the left hand side is positive and bounded is related to the fact that the Green function grows linearly away from the boundary.

In Lipschitz domains, one cannot do this computation directly and needs to proceed by approximation; see [163, Chapter 10]. Then, one can interpret Dahlberg's theorem as the Green function being linear at the boundary in a certain averaged sense.

To have a more precise understanding of the pointwise behaviour of harmonic functions, one needs the boundary Harnack. The immediate benefit of the boundary Harnack is that to understand the local properties of any harmonic function near the boundary, it is enough to understand the Green function (with any pole far away from the boundary, thanks to the boundary Harnack again).

Probability

The boundary Harnack inequality has an interpretation in terms of Brownian motions, and has been proved with probabilistic techniques [21]. In the language of probability, the boundary Harnack reads as follows.

Corollary. *Let Ω be a domain where the boundary Harnack holds, and assume that the origin belongs to $\partial\Omega$. Let $x_0 \in \Omega \cap B_{1/2}$, and let E_1 and E_2 be compact subsets of $\Omega \cap \partial B_1$. Then, for all $x \in \Omega \cap B_{1/2}$,*
$$C^{-1}\frac{\mathbb{P}(X_{x_0} \in E_1)}{\mathbb{P}(X_{x_0} \in E_2)} \leq \frac{\mathbb{P}(X_x \in E_1)}{\mathbb{P}(X_x \in E_2)} \leq C\frac{\mathbb{P}(X_{x_0} \in E_1)}{\mathbb{P}(X_{x_0} \in E_2)},$$
where X_x is the exit point of a Brownian motion in $\Omega \cap B_1$ starting at x.

This comes from the fact that $u_i(x) = \mathbb{P}(X_x \in E_i)$ are positive, harmonic in $\Omega \cap B_1$, and vanish on $\partial\Omega \cap B_1$.

The boundary Harnack also implies that the Martin boundary can be identified with the topological boundary for Lipschitz domains.

Background

Boundary Harnack inequalities have been around since the 70s, and there are many examples of such results. Here we provide a list, by no means exhaustive, focusing on local elliptic and parabolic equations, and in domains where the Hopf-Oleinik lemma does not hold.

Elliptic boundary Harnack

The first proof of the boundary Harnack for harmonic functions in Lipschitz domains is due to Kemper [125].

Since then, there have been several extensions of the result. Concerning more general operators, divergence form equations were studied by Caffarelli, Fabes, Mortola and Salsa, while the non-divergence case was covered by Fabes, Garofalo, Marin-Malave and Salsa [41, 84].

On the other hand, the boundary Harnack was one of the motivations for Jerison and Kenig to define NTA domains (non tangentially accessible domains) as the most reasonably general class where the result holds [122]. Bass and Burdzy also proved a similar result for non-divergence operators in Hölder domains using techniques from probability theory [22].

Parabolic boundary Harnack

The main difficulty that appears in the parabolic problem with respect to the elliptic case is the presence of a waiting time in the interior Harnack.

Theorem ([197])**.** *Let $\mathcal{L}$ be a non-divergence form operator as in (2), and let u be a solution to $u_t - \mathcal{L}u = 0$ in Q_r, with $u \geq 0$.*

Then,

$$\sup_{Q_{r/2}\left(0,-\frac{r^2}{2}\right)} u \leq C \inf_{Q_{r/2}} u,$$

where C depends only on the dimension and ellipticity constants.

The first proof of the boundary Harnack for parabolic equations was also by Kemper [126]. The extension to divergence form operators (in Lipschitz cylinders) is due to Fabes, Garofalo, and Salsa [173, 85].

The Hölder continuity of the quotient of positive solutions was shown for the first time by Athanasopoulos, Caffarelli, and Salsa [10]. Note that, in contrast to the elliptic setting, where this was an almost automatic consequence of the L^∞ boundary Harnack, for parabolic equations the $C^{0,\alpha}$ boundary Harnack appeared more than two decades later than the boundedness of the quotient. This result was later extended to non-divergence form equations by Fabes, Safonov and Yuan [86, 87].

Moreover, more general domains have also been studied. Hölder cylindrical domains were treated by Bass and Burdzy with probabilistic techniques [21], Hoffman, Lewis, and Nyström studied unbounded parabolically Reifenberg flat domains [117], and Petrosyan and Shi considered parabolic slit domains, i.e. domains defined as a cylinder minus a Lipschitz graph in a hyperplane, that appear naturally in the parabolic thin obstacle problem [161].

Recent developments

In the last few years, De Silva and Savin have given short and unified analytic proofs of the boundary Harnack based on scaling arguments and the comparison principle. The results apply to divergence and non-divergence form equations, and they hold in Lipschitz and Hölder domains. For elliptic equations, they proved the boundedness and Hölder regularity of the quotient of positive solutions up to the boundary [73, 74], while in the parabolic case they only proved the L^∞ estimate [75].

Another active area of research concerns higher order boundary Harnack inequalities. The first version of the result was presented by De Silva and Savin in [71], with an application to show C^∞ regularity of the free boundary in obstacle problems with smooth obstacle. The parabolic higher order boundary Harnack was first proved in $C^{k,\alpha}$ domains by Banerjee and Garofalo [13], and extended to C^1 domains by Kukuljan [140].

An alternative approach to higher order boundary Harnack inequalities is based on the fact that if u and v are solutions to elliptic equations, then u/v solves a degenerate elliptic equation and one can study its regularity. In this line of research, we find the contributions by Terracini, Tortone, Vita, Zhang, Jeon and Dong [193, 202, 121, 78].

Finally, there have been important recent advances in boundary Harnack inequalities for equations with right-hand side. The first result of this kind is due to Allen and Shahgholian [3], where they considered divergence form operators in Lipschitz domains and the right-hand side was in a weighted L^∞ space. Together with Kriventsov, they extended the result to a very general class of equations, including fully nonlinear elliptic and p-Laplace type equations [2]. The other main contribution are the results of this Thesis, which we will explain in the following.

Results of the thesis (Part II)

The second part of this Thesis is devoted to the study of boundary Harnack inequalities for elliptic and parabolic equations with right-hand side.

Chapter 4 deals with elliptic equations in non-divergence and divergence form, and in Chapter 5 we study parabolic equations in non-divergence form.

The boundary Harnack for elliptic equations with right-hand side

In Chapter 4 we prove a boundary Harnack inequality in flat Lipschitz domains for elliptic equations in non-divergence or divergence form, with a small right-hand side in L^q.

Theorem. *Let $q > n$ and $\mathcal{L}$ as in (2), or (1) with continuous coefficients. There exist small constants $c_0 > 0$ and $L_0 > 0$ such that the following holds.*

Let Ω be a Lipschitz domain with Lipschitz constant $L < L_0$. Let u and v be positive solutions of

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \cap B_1 \\ u = 0 & \text{on } \partial\Omega \cap B_1 \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}v = g & \text{in } \Omega \cap B_1 \\ v = 0 & \text{on } \partial\Omega \cap B_1, \end{cases}$$

with

$$\|f\|_{L^q(B_1)} \leq c_0, \quad \|g\|_{L^q(B_1)} \leq c_0.$$

Additionally, assume that $v(e_n/2) = u(e_n/2) = 1$. Then,

$$u \leq Cv \quad \text{in} \quad B_{1/2},$$

and
$$\left\| \frac{u}{v} \right\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega}\cap B_{1/2})} \leq C.$$
The constants C, c_0, L_0 and $\alpha > 0$ depend only on q, the dimension, ellipticity constants, and the modulus of continuity of the coefficients of $\mathcal{L}$, when applicable.

The result is sharp, in the following sense.

Theorem.

- *If the Lipschitz constant of the domain is large, the theorem fails, even for a right-hand side in L^∞ and $\mathcal{L} = \Delta$.*
- *If the right-hand side belongs to L^q with $q \leq n$, the theorem fails, regardless of the norm and the Lipschitz constant of the domain.*
- *There are examples of divergence form equations with discontinuous coefficients with a right-hand side in L^∞ where the theorem fails, even in a half-space.*

The proof is done by comparison and scaling arguments following the strategy of [73], with the addition of some fine estimates on the growth of $\mathcal{L}$-harmonic functions near zero Dirichlet conditions, that can be summarized as:

For all $\varepsilon > 0$, there exists a sufficiently small $L > 0$, such that for all Lipschitz domains with constant L, any positive harmonic function that vanishes on the boundary is bounded below by $cd^{1+\varepsilon}$, where d is the distance to the boundary.

This kind of boundary nondegeneracy result acts as the replacement of the Hopf-Oleinik lemma for flat Lipschitz domains, and it is crucial to be able to absorb the perturbation introduced by the right-hand side.

The main consequence of our new boundary Harnack is the $C^{1,\alpha}$ regularity of free boundaries in the fully nonlinear obstacle problem.

Corollary. *Let u be a solution to*
$$\begin{cases} F(D^2u, x) &= f\chi_{\{u>0\}} \\ u &\geq 0 \end{cases}$$
Assume as well:

- *F is uniformly elliptic and $F(0, x) \equiv 0$.*
- *F is convex and $\mathcal{C}^1$ in the first variable, and $W^{1,q}$ in the second variable for some $q > n$.*
- *$f \in W^{1,q}$ for some $q > n$, and $f \geq \tau_0 > 0$.*

Then, if the origin is a regular free boundary point, the free boundary is a $\mathcal{C}^{1,\alpha}$ graph in B_r for some small $r > 0$ and $\alpha > 0$.

In this result, the notion of regular free boundary point needs to be understood as having a regular blow-up.

We also prove an analogous result for slit domains, that implies the $C^{1,\alpha}$ regularity of free boundaries in the fully nonlinear thin obstacle problem.

The boundary Harnack for parabolic equations with right-hand side

Chapter 5 is dedicated to proving a boundary Harnack inequality in flat Lipschitz domains for parabolic equations in non-divergence form, with a small right-hand side in L^q.

The main result is very similar to the main result of Chapter 4, but for the parabolic case. We will state it for the heat equation for ease of read, but it also holds for non-divergence form parabolic equations with bounded measurable coefficients.

Theorem. *Let $m \in (0,1]$ and $q > n+2$. Then, for every $\gamma \in \left(0, 1 - \frac{n+2}{q}\right)$, there exists $c_0 > 0$, only depending on the dimension, q, and γ, such that the following holds.*

Let Ω be a parabolic Lipschitz domain with Lipschitz constant c_0. Let u and v be positive solutions to

$$\left\{\begin{array}{rcll} u_t - \Delta u & = & f & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \partial\Omega \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rcll} v_t - \Delta v & = & g & \text{in} \quad \Omega \\ v & = & 0 & \text{on} \quad \partial\Omega, \end{array}\right.$$

and assume that u and v are normalized so that $\|u\|_{L^\infty(Q_1)} = \|v\|_{L^\infty(Q_1)} = 1$, $v\left(\frac{e_n}{2}, -\frac{3}{4}\right) \geq m$, $\|f\|_{L^q(Q_1)} \leq 1$, and $\|g\|_{L^q(Q_1)} \leq c_0 m$. Then,

$$\left\|\frac{u}{v}\right\|_{C_p^{0,\gamma}(\Omega\cap Q_{1/2})} \leq C,$$

where C depends only on q, m, γ, and the dimension.

In the parabolic setting, there are analogous applications to prove the $C^{1,\alpha}$ regularity of free boundaries at regular points in parabolic obstacle problems.

We also prove a similar result for the heat equation in parabolic slit domains, with an application to the regularity of the free boundary in the parabolic thin obstacle problem.

Another strength of the result is obtaining the optimal Hölder space for the regularity of the quotient, which is completely new, even for harmonic functions. In this particular case, we get that, for every $\varepsilon > 0$, there exists $L > 0$ such that the quotient of harmonic functions is $C^{1-\varepsilon}$ up to the boundary in Lipschitz domains with constant L. This bridges the gap from the known $C^{0,\alpha}$ regularity with small α for Lipschitz domains with a possibly big Lipschitz constant, and higher order boundary Harnack results.

Despite the fact that the statement and the consequences are analogous to the elliptic framework, the techniques of the proof are quite different. In the parabolic setting, the Hölder regularity of the quotient does not follow easily from the L^∞ estimate, and then one needs to prove directly the $C^{0,\gamma}$ estimate.

The proof uses some ingredients of the elliptic problem, like the scaling and comparison arguments, and also a two-sided growth estimate that reads as

For all $\varepsilon > 0$, there exists a sufficiently small $L > 0$, such that for all parabolic Lipschitz domains with constant L, any positive caloric function u that vanishes on the boundary satisfies

$$cd^{1+\varepsilon} \leq u \leq Cd^{1-\varepsilon},$$

where d is the distance to the boundary.

The key idea, that is new from the parabolic setting, is using a contradiction-compactness argument (also called blow-up argument) inspired in some proofs of boundary Harnack inequalities for smoother domains [167, 140].

The argument works as in the proof of Schauder estimates by blow-up. We suppose that our estimate does not hold, and then we get a sequence of solutions to a PDE with very specific growth properties. Since they are solutions, we can prove some regularity, and from the regularity we deduce compactness of a subsequence via Arzelà-Ascoli. Then, the limit of those solutions solves a limit problem in a simpler domain (in our case, in a half-space), and then we can classify the possible limits with a Liouville theorem. The contradiction comes from the fact that none of the possibilities allowed by the Liouville theorem can be a limit of our sequence.

We exploit this contradiction-compactness argument twice. First, to prove that for flat Lipschitz domains, there exists a special solution that satisfies the growth estimate *at all scales at the same time*, that is,

For all $\varepsilon > 0$, there exists a sufficiently small $L > 0$, such that for all parabolic Lipschitz domains with constant L, there exists a caloric function $\varphi > 0$ such that

$$c\left(\frac{r}{R}\right)^{1+\varepsilon} \leq \frac{\|\varphi\|_{L^\infty(B_r\times(-r^2,0))}}{\|\varphi\|_{L^\infty(B_R\times(-R^2,0))}} \leq C\left(\frac{r}{R}\right)^{1-\varepsilon},$$

for all $0 < r < R < 1$.

Then, by another contradiction-compactness argument, we see that

For all $\varepsilon > 0$, there exists a sufficiently small $L > 0$, such that for all parabolic Lipschitz domains with constant L, any caloric function u satisfies

$$\|u - C(r)\varphi\|_{L^\infty(B_r\times(-r^2,0))} \leq Cr^{1+\alpha},$$

where φ is the special solution defined above.

From the expansion, it follows that $u/\varphi \in C^{\alpha-\varepsilon}$, which in turns implies the boundary Harnack. A posteriori, we see that all solutions satisfy the same growth condition as the special solution.

Chapter 4

New boundary Harnack inequalities with right hand side

We prove new boundary Harnack inequalities in Lipschitz domains for equations with a right hand side. Our main result applies to non-divergence form operators with bounded measurable coefficients and to divergence form operators with continuous coefficients, whereas the right hand side is in L^q with $q > n$. Our approach is based on the scaling and comparison arguments of [73], and we show that all our assumptions are sharp.

As a consequence of our results, we deduce the $\mathcal{C}^{1,\alpha}$ regularity of the free boundary in the fully nonlinear obstacle problem and the fully nonlinear thin obstacle problem.

4.1 Introduction

4.1.1 Background

The boundary Harnack inequality states that all positive harmonic functions with zero boundary condition are locally comparable as they approach the boundary, under appropriate assumptions on the domain. More precisely, if u and v are positive harmonic functions in Ω that vanish on $\partial\Omega$, then

$$C^{-1} \le \frac{u}{v} \le C,$$

with C depending on the dimension and $u(p)/v(p)$ for a fixed interior point p.

Notice that such a result is most relevant in domains that are less regular than $\mathcal{C}^{1,\mathrm{Dini}}$, because otherwise the Hopf lemma combined with the $\mathcal{C}^1(\overline{\Omega})$ regularity of the solutions yields the same conclusion, see for example [147].

The boundary Harnack inequality is known to be true for a broad class of domains and for solutions of more general elliptic equations. The classical case for harmonic functions was first proved by Kemper in Lipschitz domains in [125]. Operators in divergence form were first considered by Caffarelli, Fabes, Mortola and Salsa in [41] in Lipschitz domains, while the case of operators in non-divergence form was treated in [84] by Fabes, Garofalo, Marin-Malave and Salsa. Jerison and Kenig extended the same result to NTA domains in the case of divergence form operators in [122]. On the other hand, the case of non-divergence operators in Hölder domains with $\alpha > 1/2$ was treated with probabilistic techniques in [22] by Bass and Burdzy.

Recently, De Silva and Savin found a simple and unified proof of all these previous results in [73].

Besides, Allen and Shahgholian recently proved the boundary Harnack for divergence form equations **with right hand side** in Lipschitz domains [3], under appropriate assumptions on the operator, the right hand side and the domain. In particular, in the case of the Laplacian, their result implies that if the L^∞ norm of the right hand side and the Lipschitz constant of the domain are small enough, then the boundary Harnack inequality still holds. This enables using the classical proof in [36] due to Caffarelli (see also [160, Section 6.2] or [97, Section 5.4]) of the regularity of the free boundary in the obstacle problem $\Delta u = \chi_{\{u>0\}}$ in the more general case $\Delta u = f\chi_{\{u>0\}}$, with f Lipschitz; see [3, Section 1.4.2].

Here, we extend such boundary Harnack inequality to **non-divergence** equations with possibly **unbounded** right hand side in L^q, with $q > n$. (This was only known in $\mathcal{C}^{1,1}$ domains [189, 190].) This allows us to use the classical proof of the free boundary regularity in the obstacle problem $\Delta u = f\chi_{\{u>0\}}$ to the case $f \in W^{1,q}$, and can also be applied to fully nonlinear free boundary problems of the form

$$F(D^2u) = f\chi_{\{u>0\}} \quad \text{or} \quad \begin{cases} F(D^2v) & = & 0 & \text{in } \{v > \varphi\} \\ F(D^2v) & \leq & 0 & \\ v & \geq & \varphi. & \end{cases} \tag{4.1}$$

Moreover, we also establish a boundary Harnack for equations with a right hand side in *slit domains*, and use it to establish the $\mathcal{C}^{1,\alpha}$ regularity of the free boundary in the fully nonlinear thin obstacle problem, a question left open in [168].

4.1.2 Setting

In the following, $\mathcal{L}$ will denote either a non-divergence form elliptic operator with **bounded measurable** coefficients,

$$\mathcal{L}u = \mathrm{Tr}(A(x)D^2u), \quad \text{with} \quad \lambda I \leq A(x) \leq \Lambda I, \tag{4.2}$$

with $0 < \lambda \leq \Lambda$, or a divergence form elliptic operator with **continuous** coefficients,

$$\mathcal{L}u = \mathrm{Div}(A(x)\nabla u), \quad \text{with} \quad \lambda I \leq A(x) \leq \Lambda I \quad \text{and} \quad A \in \mathcal{C}^0, \tag{4.3}$$

where A has modulus of continuity σ, and $0 < \lambda \leq \Lambda$.

We will consider Lipschitz domains of the following form, where B_1' is the unit ball of $\mathbb{R}^{n-1}$.

Definition 4.1.1. We say Ω is a Lipschitz domain with Lipschitz constant L if Ω is the epigraph of a Lipschitz function $g : B_1' \to \mathbb{R}$, with $g(0) = 0$:

$$\Omega = \Big\{(x', x_n) \in B_1' \times \mathbb{R} \quad \text{such that} \quad x_n > g(x')\Big\}, \quad \|g\|_{\mathcal{C}^{0,1}} = L.$$

4.1.3 Main results

We present here our new boundary Harnack inequality.

We emphasize that the following result applies to both non-divergence and divergence form operators, and that the only regularity assumption on the coefficients is the continuity of $A(x)$

in case of divergence-form operators. Throughout the paper, when we say L^n-viscosity or weak solutions, we refer to L^n-viscosity solutions in the case of non-divergence form operators (4.2), and to weak solutions in the case of divergence form operators (4.3).

Theorem 4.1.2. *Let $q > n$ and $\mathcal{L}$ as in (4.2) or (4.3). There exist small constants $c_0 > 0$ and $L_0 > 0$ such that the following holds.*

Let Ω be a Lipschitz domain as in Definition 4.1.1, with Lipschitz constant $L < L_0$. Let u and $v > 0$ be solutions of

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \cap B_1 \\ u = 0 & \text{on } \partial\Omega \cap B_1 \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}v = g & \text{in } \Omega \cap B_1 \\ v = 0 & \text{on } \partial\Omega \cap B_1, \end{cases}$$

in the L^n-viscosity or the weak sense, with

$$\|f\|_{L^q(B_1)} \leq c_0, \quad \|g\|_{L^q(B_1)} \leq c_0. \tag{4.4}$$

Additionally, assume that $v(e_n/2) \geq 1$ and either $u > 0$ and $u(e_n/2) \leq 1$, or $\|u\|_{L^p(B_1)} \leq 1$ for some $p > 0$.

Then,

$$u \leq Cv \quad \text{in} \quad B_{1/2},$$

and

$$\left\| \frac{u}{v} \right\|_{C^{0,\alpha}(\overline{\Omega} \cap B_{1/2})} \leq C.$$

The constants C, c_0, L_0 and $\alpha > 0$ depend only on the dimension, q, λ, Λ, as well as p and σ, when applicable.

Remark 4.1.3. All the hypotheses of the theorem are optimal in the following sense:

- If the Lipschitz constant L_0 of the domain is not small, the theorem fails, even for $q = \infty$ and for $\mathcal{L} = \Delta$.
- If $q = n$, the theorem fails for any $c_0 > 0$ and any $L_0 > 0$, even for $\mathcal{L} = \Delta$.
- The result fails in general for operators in divergence form with bounded measurable coefficients.

We provide counterexamples to plausible extensions in this sense in Section 4.6.

When the two functions are positive, we recover the standard symmetric formulation of the boundary Harnack.

Corollary 4.1.4. *Let $q > n$ and $\mathcal{L}$ as in (4.2) or (4.3). There exist small constants $c_0 > 0$ and $L_0 > 0$ such that the following holds. Let Ω be a Lipschitz domain as in Definition 4.1.1, with Lipschitz constant $L < L_0$. Let u, v be positive solutions of*

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \cap B_1 \\ u = 0 & \text{on } \partial\Omega \cap B_1 \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}v = g & \text{in } \Omega \cap B_1 \\ v = 0 & \text{on } \partial\Omega \cap B_1, \end{cases}$$

in the L^n-viscosity or the weak sense, with f and g satisfying (4.4).

Assume u, v are normalized in the sense that $u(e_n/2) = v(e_n/2) = 1$. Then,

$$C^{-1} \leq \frac{u}{v} \leq C \quad \text{in} \quad B_{1/2},$$

and

$$\left\| \frac{u}{v} \right\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega} \cap B_{1/2})} \leq C.$$

The positive constants C, c_0, L_0 and α depend only on the dimension, q, λ, Λ, as well as σ, when applicable.

4.1.4 Applications to obstacle problems

The boundary Harnack inequality is the technical tool that allows us to prove $\mathcal{C}^{1,\alpha}$ regularity of the free boundary once we know it is Lipschitz in the classical obstacle problem with constant right hand side [97, Section 5.6] and in the thin obstacle problem with zero obstacle [91, Section 5].

The functions to which we apply the boundary Harnack are derivatives of the solution to the free boundary problem. Hence, if the original free boundary problem is the classical obstacle problem,

$$\begin{cases} \Delta u &= f\chi_{\{u>0\}} \\ u &\geq 0, \end{cases} \tag{4.5}$$

the derivatives of u are solutions of

$$\begin{cases} \Delta(\partial_\nu u) &= \partial_\nu f & \text{in } \{u > 0\} \\ \partial_\nu u &= 0 & \text{on } \partial\{u > 0\}, \end{cases}$$

and we can apply the boundary Harnack of Allen and Shahgholian if $f \in W^{1,\infty}$ (Lipschitz), or our new Theorem 4.1.2 if $f \in W^{1,q}$ with $q > n$.

In the fully nonlinear setting (4.1), the derivatives of the solution satisfy a linear equation in non-divergence form,

$$\mathcal{L}(\partial_\nu u) = g \quad \text{in} \quad \{u > 0\},$$

with bounded measurable coefficients $A(x)$, and then having our new boundary Harnack for non-divergence operators proves useful to deduce results on the regularity of the free boundary.

It is well known that the free boundary may exhibit singularities. Hence, we need to introduce the notion of a regular point.

Definition 4.1.5. Let x_0 be a free boundary point for the classical obstacle problem, i.e. $x_0 \in \partial\{u > 0\}$ for a solution of (4.5). We say that x_0 is a regular free boundary point if there exists $r_k \downarrow 0$ such that

$$\frac{u(r_k x)}{r_k^2} \to \frac{\gamma}{2}(x \cdot e)_+^2 \quad \text{in } \mathcal{C}^1_{\text{loc}}(\mathbb{R}^n)$$

for some $\gamma > 0$ and $e \in \mathbb{S}^{n-1}$.

Our next application was already known by using perturbative arguments with slightly weaker assumptions [23]. We include this result to illustrate the arguments that we will use in the fully nonlinear problems in a more easily readable setting.

Corollary 4.1.6. *Let u be a solution of (4.5) with $f \geq c_0 > 0$ in $W^{1,q}(B_1)$, with $q > n$, and assume the origin is a regular free boundary point in the sense of Definition 4.1.5.*

Then, the free boundary $\Gamma = \partial\{u > 0\}$ is locally a $\mathcal{C}^{1,\alpha}$ graph at 0.

The fully nonlinear obstacle problem can be presented in at least two different formulations. The following one was studied by Lee in [143].

$$\begin{cases} F(D^2 v) & \leq & 0 \\ v & \geq & \varphi \\ F(D^2 v) & = & 0 \quad \text{in} \quad \{v > \varphi\}. \end{cases} \tag{4.6}$$

Here, we impose the following conditions:

- F is uniformly elliptic.
- $F(D^2\varphi) \leq -\tau_0 < 0$.
- $\varphi \in \mathcal{C}^\infty$.

Then, under these hypotheses, $v \in \mathcal{C}^{1,1}$ and the free boundary is $\mathcal{C}^{1,\alpha}$ at regular points. For our purposes, we will say a free boundary point is regular in the sense of Definition 4.1.5, as in the classical obstacle problem.

More generally, one can study problems of the form

$$\begin{cases} F(D^2 u, x) = f\chi_{\{u>0\}} \\ u \geq 0. \end{cases} \tag{4.7}$$

This is a generalization of problem (4.6). Indeed, if we define $u = v - \varphi$, then

$$\tilde{F}(D^2 u, x) := F(D^2 u + D^2\varphi) - F(D^2\varphi) = -F(D^2\varphi) =: f(x) \quad \text{in} \quad \{u > 0\}.$$

This fully nonlinear obstacle problem (and more general ones without the sign condition on u) has been further studied by Lee, Shahgholian, Figalli, and more recently by Indrei and Minne in [144, 105, 120]. They proved that if F is convex, f is Lipschitz and $f \geq \tau_0 > 0$, the free boundary $\partial\Omega$ is $\mathcal{C}^1$ at regular points.

As a consequence of our new boundary Harnack inequality, we extend their result for (4.7) in two ways. We lower the Lipschitz regularity required for f to $W^{1,q}$ with $q > n$, and we prove $\mathcal{C}^{1,\alpha}$ regularity of the free boundary instead of $\mathcal{C}^1$.

Corollary 4.1.7. *Let u be a solution of (4.7). Assume as well:*

(H1) F is uniformly elliptic and $F(0, x) = 0$ for all $x \in \Omega$.

(H2) F is convex and $\mathcal{C}^1$ in the first variable, and $W^{1,q}$ in the second variable for some $q > n$.

(H3) $f \in W^{1,q}$ for some $q > n$, and $f \geq \tau_0 > 0$.

Then, if the origin is a regular free boundary point in the sense of Definition 4.1.5, the free boundary is a $\mathcal{C}^{1,\alpha}$ graph in B_r for some small $r > 0$ and $\alpha > 0$.

4.1.5 Thin obstacle problems

The thin obstacle problem, also known as the Signorini problem, is a classical free boundary problem that admits several formulations, see [91] for a nice introduction to the topic. One can write the problem as the following, given an obstacle φ defined on $\{x_n = 0\}$:

$$\begin{cases} \Delta v & \le & 0 & \text{in } B_1 \\ v & \ge & \varphi & \text{on } B_1 \cap \{x_n = 0\} \\ \Delta v & = & 0 & \text{in } B_1 \setminus \{(x', 0) : v(x', 0) = \varphi(x')\}. \end{cases} \tag{4.8}$$

The first results on regularity of the solution v were established in the seventies, in particular it was proved in [35] that $v \in \mathcal{C}^{1,\alpha}$ for a small $\alpha > 0$. Free boundary regularity remained open for quite some time, until the first free boundary regularity result, [11], establishing that the free boundary is $\mathcal{C}^{1,\alpha}$ at *regular* points when $\varphi \equiv 0$. Further results have been obtained in [133, 72] among others, proving that the free boundary is real analytic at regular points provided that φ is analytic.

Consider now the fully nonlinear thin obstacle problem.

$$\begin{cases} F(D^2 v) & \le & 0 & \text{in } B_1 \\ v & \ge & \varphi & \text{on } B_1 \cap \{x_n = 0\} \\ F(D^2 v) & = & 0 & \text{in } B_1 \setminus \{(x', 0) : v(x', 0) = \varphi(x')\}, \end{cases} \tag{4.9}$$

where F is uniformly elliptic, convex and $F(0) = 0$.

Milakis and Silvestre proved in [152] that solutions u are $\mathcal{C}^{1,\alpha}$ in the symmetric case (even functions with respect to x_n). More recently, Fernández-Real extended the result to the non-symmetric case in [90]. The first result on free boundary regularity is due to the first author and Serra [168], where they proved the $\mathcal{C}^1$ regularity of the free boundary near regular points. Here, we will prove for the first time that the free boundary is actually $\mathcal{C}^{1,\alpha}$.

To do this, we need to adapt Theorem 4.1.2 to the case of slit domains. We present here a simplified version, see Section 4.4 for a more general result.

Theorem 4.1.8. *Let $q > n$ and let $\mathcal{L}$ be as in (4.2). There exists small $c_0 > 0$ such that the following holds.*

Let $\Omega = B_1 \setminus K$ with K a closed subset of $\{x_n = 0\}$. Let

$$\Omega^+ = \Omega \cap \{x_n \ge 0\} \quad \text{and} \quad \Omega^- = \Omega \cap \{x_n \le 0\}.$$

Let u and $v > 0$ be L^n-viscosity solutions of

$$\begin{cases} \mathcal{L}u & = & f & \text{in } B_1 \setminus K \\ u & = & 0 & \text{on } K \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}v & = & g & \text{in } B_1 \setminus K \\ v & = & 0 & \text{on } K, \end{cases}$$

with f and g satisfying (4.4). Assume in addition that $v(e_n/2) \ge 1$, $v(-e_n/2) \ge 1$, and either $u > 0$ in $B_1 \setminus K$ and $\max\{u(e_n/2), u(-e_n/2)\} \le 1$, or $\|u\|_{L^p(B_1)} \le 1$ for some $p > 0$. Then,

$$u \le Cv \quad \text{in} \quad B_{1/2} \setminus K,$$

and

$$\left\| \frac{u}{v} \right\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega^\pm} \cap B_{1/2})} \le C.$$

The positive constants C, c_0, and α depend only on the dimension, q, λ, Λ, as well as p, when applicable.

Using this new boundary Harnack, we can prove the following.

Corollary 4.1.9. *Assume that* 0 *is a regular free boundary point for (4.9) in the sense of [168], with* $F \in \mathcal{C}^1$ *and* $\varphi \in W^{3,q}$ *for some* $q > n$*. Then, there exists* $\rho > 0$ *such that the free boundary is a* $\mathcal{C}^{1,\alpha}$ *graph in* $B_\rho \cap \{x_n = 0\}$*.*

This is new, even when $\varphi \in \mathcal{C}^\infty$. The higher regularity of the free boundary remains a challenging open question.

4.1.6 Plan of the paper

The paper is organized as follows.

In Section 4.2, we recall some classical results and tools, such as the ABP estimate and the weak Harnack inequality. Then, in Section 4.3 we prove our new boundary Harnack inequality for elliptic equations with right hand side, Theorem 4.1.2, by scaling and barrier arguments. Section 4.4 is devoted to adapting the result to slit domains. In Section 4.5, we prove the $\mathcal{C}^{1,\alpha}$ regularity of the free boundary in the fully nonlinear obstacle problem, Corollary 4.1.7, and in the fully nonlinear thin obstacle problem, Corollary 4.1.9. Finally, in Section 4.6, we present two counterexamples that show the sharpness of our new boundary Harnack and in Section 4.7 we introduce a Hopf lemma for equations with right hand side.

4.2 Preliminaries

In this section we recall some classical tools and results that will be used throughout the paper. We will denote

$$\mathcal{M}^-(D^2u) := \inf_{\lambda I \le A \le \Lambda I} \mathrm{Tr}(AD^2u), \quad \mathcal{M}^+(D^2u) := \sup_{\lambda I \le A \le \Lambda I} \mathrm{Tr}(AD^2u)$$

the Pucci extremal operators, see [37] or [97] for their properties.

4.2.1 L^n-viscosity and weak solutions

In this work we are considering linear elliptic equations of the form $\mathcal{L}u = f$, with $f \in L^q$, with $q \ge n$. The most appropriate notion of solutions for a divergence form equation are the well-known weak solutions.

For the non-divergence form case, one could consider *strong* ($W^{2,n}_{\mathrm{loc}}$, solving the PDE in the a.e. sense) solutions, but all the arguments of the proof are equally viable for L^n-viscosity solutions, which are more general. We present the minimal definition for the linear case.

Definition 4.2.1 ([38])**.** Let $u \in \mathcal{C}(\Omega)$, $f \in L^n_{\mathrm{loc}}(\Omega)$ and $\mathcal{L}$ in non-divergence form. We say u is a L^n-viscosity subsolution (resp. supersolution) if, for all $\varphi \in W^{2,n}_{\mathrm{loc}}(\Omega)$ such that $u - \varphi$ has a local maximum (resp. minimum) at x_0,

$$\operatorname{ess}\liminf_{x\to x_0} \mathcal{L}\varphi - f \le 0$$

$$(\text{resp. } \operatorname{ess}\limsup_{x\to x_0} \mathcal{L}\varphi - f \ge 0).$$

We will say equivalently that u is a solution of $\mathcal{L}u \leq (\geq) f$. When u is both a subsolution and a supersolution, we say u is a solution and write $\mathcal{L}u = f$.

L^n-viscosity solutions coincide with strong, viscosity or even classical solutions when they have the required regularity, and satisfy the maximum and comparison principles, but are more flexible, for example, allowing to compute limits under some reasonable hypotheses, and are thus preferred in some contexts.

Throughout this paper, the Dirichlet boundary conditions must be understood in the pointwise sense when we are dealing with L^n-viscosity solutions, and in the trace sense when we are dealing with weak solutions.

4.2.2 Interior estimates

The Alexandroff-Bakelmann-Pucci estimate is one of the main tools in regularity theory for non-divergence form elliptic equations. We refer to [37, Theorem 3.2] and [38, Proposition 3.3] for the full details and a proof.

Theorem 4.2.2 (ABP Estimate)**.** *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain. Let $\mathcal{L}$ be a non-divergence form operator as in (4.2) and let $u \in \mathcal{C}(\overline{\Omega})$ satisfy $\mathcal{L}u \geq f$ in the L^n-viscosity sense, with $f \in L^n(\Omega)$. Assume that u is bounded on $\partial\Omega$.*

Then,

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + C \operatorname{diam}(\Omega) \|f\|_{L^n(\Omega)}$$

with C only depending on the dimension, λ and Λ.

In the case of divergence form equations, the global boundedness of weak solutions is known in more generality. For our purposes, it is sufficient to consider the case $p = n$.

Theorem 4.2.3 ([112, Theorem 8.16])**.** *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain. Let $\mathcal{L}$ be a divergence form operator as in (4.3) and let $u \in \mathcal{C}(\overline{\Omega})$ be a weak solution of $\mathcal{L}u \geq f$, with $f \in L^p(\Omega)$, $p > n/2$. Assume that u is bounded on $\partial\Omega$.*

Then,

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + C\|f\|_{L^p(\Omega)}$$

with C only depending on the dimension, $|\Omega|$, p, λ and Λ.

We will need the two estimates that are classically combined to obtain the Krylov-Safonov Harnack inequality. The first one is the following weak Harnack inequality, valid for L^n-viscosity solutions of non-divergence form equations. We refer to [194, Theorem 2] and [135, Theorem 4.5].

Theorem 4.2.4 (Weak Harnack inequality)**.** *Let $\mathcal{L}$ be a non-divergence form operator as in (4.2). Let u satisfy $\mathcal{L}u \leq 0$ in Ω in the L^n-viscosity sense and let $B_R(y) \subset \Omega$. Then, for all $\sigma < 1$,*

$$\|u\|_{L^p(B_{\sigma R})} \leq C \inf_{B_{\sigma R}} u,$$

where p and C are positive and depend only on the dimension, σ and Λ/λ.

Now, combining this theorem with the ABP estimate, applied to the function $1-u$, we obtain the following result. This is also valid for divergence form equations, and sometimes known as De Giorgi oscillation lemma in that setting. The case with $f = 0$ is found in [47, Theorem 11.2], and we can extend it easily to the general case using Theorem 4.2.3.

Corollary 4.2.5. *Let $\mathcal{L}$ be as in (4.2) or (4.3). Let $r \in (0,1]$, $u \le 1$, $\mathcal{L}u \ge f$ in B_r, in the L^n-viscosity or the weak sense, with $f \in L^n(B_r)$. Assume $|\{u \le 0\}| \ge \eta|B_r| > 0$, and that $\|f\|_{L^n(B_r)} \le \delta(\eta)$. Then,*

$$\sup_{B_{r/2}} u \le 1 - \gamma(\eta),$$

where $\delta(\eta) > 0$ and $\gamma(\eta) \in (0,1)$ depend only on the dimension, λ, Λ and η.

The second estimate is the upper bound in Harnack inequality, also valid for L^n-viscosity solutions of non-divergence form equations [194, Theorem 1], [136] and weak solutions of divergence form equations [70, 146]. In the divergence form case, we can add the right hand side using Theorem 4.2.3.

Theorem 4.2.6 (L^∞ bound for subsolutions)**.** *Let $p > 0$ and let $\mathcal{L}$ be as in (4.2) or (4.3). Let $\mathcal{L}u \ge f$ in B_1, in the L^n-viscosity or the weak sense. Then,*

$$\sup_{B_{1/2}} u \le C_p(\|u\|_{L^p(B_1)} + \|f\|_{L^n(B_1)}),$$

where $C_p > 0$ depends only on the dimension, p, λ and Λ.

4.3 Proof of Theorem 4.1.2

4.3.1 Nondegeneracy

To study solutions of $\mathcal{L}u = f$ in a Lipschitz domain it is useful to know their behaviour in a cone. In this first part of the proof we show that, much like solutions of elliptic equations with zero Dirichlet boundary conditions separate linearly from the boundary of the domain in domains with the interior ball condition (Hopf lemma), the solutions of elliptic equations with zero Dirichlet boundary conditions separate as a power of the distance at corners, and the exponent approaches 1 as the corners become wider.

Lemma 4.3.1. *Let $\mathcal{L}$ be in non-divergence form as in (4.2). Let $\beta > 1$. There exist sufficiently small $c(\beta) > 0, \eta > 0$, only depending on the dimension, β, λ and Λ, such that the following holds.*

Let u be any solution of

$$\begin{cases} \mathcal{L}u & \le & c(\beta) & \text{in } C_\eta \\ u & \ge & 1 & \text{on } \{x_n = 1\} \cap \overline{C_\eta} \\ u & \ge & 0 & \text{in } \partial C_\eta, \end{cases}$$

where C_η is the cone defined as

$$C_\eta := \{x \in \mathbb{R}^n : \eta|x'| < x_n < 1\}.$$

Then,

$$u(te_n) \ge t^\beta, \quad \forall t \in (0,1).$$

Proof. Assume without loss of generality that $\beta \in (1,2)$, because if the inequality holds for $\beta > 1$, it holds also for all $\beta' > \beta$. We will use the comparison principle with a subsolution that has the desired behaviour. Let $\varepsilon \in (0, 1/20)$ to be chosen later. Notice that $\sqrt{1+\varepsilon} - \sqrt{\varepsilon} > 4/5$. Define the subsolution φ as:

$$\varphi(x) = x_n^\beta f_\varepsilon\left(\frac{\eta|x'|}{x_n}\right), \quad f_\varepsilon(t) = \frac{\sqrt{1+\varepsilon} - \sqrt{t^2+\varepsilon}}{\sqrt{1+\varepsilon} - \sqrt{\varepsilon}}.$$

We can readily check that $\varphi(x) = 0$ for $x \in \partial C_\eta$. It is also clear that $\varphi(x) \le 1$ in $\{x_n = 1\} \cap C_\eta$, and that $\varphi > 0$ in C_η. Now, we need some estimates on f_ε and its derivatives. For $t \in [0,1)$,

$$\begin{aligned}
f_\varepsilon(t) &= \frac{\sqrt{1+\varepsilon} - \sqrt{t^2+\varepsilon}}{\sqrt{1+\varepsilon} - \sqrt{\varepsilon}} \ge \frac{\sqrt{1+\varepsilon} - t - \sqrt{\varepsilon}}{\sqrt{1+\varepsilon} - \sqrt{\varepsilon}} = 1 - \frac{t}{\sqrt{1+\varepsilon} - \sqrt{\varepsilon}} > 1 - \frac{5}{4}t \\
f_\varepsilon'(t) &= -\frac{t}{\sqrt{t^2+\varepsilon}(\sqrt{1+\varepsilon} - \sqrt{\varepsilon})} \ge \frac{-1}{\sqrt{1+\varepsilon} - \sqrt{\varepsilon}} > -\frac{5}{4} \\
|t^{-1} f_\varepsilon'(t)| &\le \frac{1}{\sqrt{\varepsilon}(\sqrt{1+\varepsilon} - \sqrt{\varepsilon})} < \frac{5}{4}\varepsilon^{-1/2} \\
f_\varepsilon'(t) &\le \frac{-t}{(t+\sqrt{\varepsilon})(\sqrt{1+\varepsilon} - \sqrt{\varepsilon})} \le \frac{-t}{1+\varepsilon} < -\frac{20}{21}t \\
|f_\varepsilon''(t)| &= \left|\frac{-\varepsilon}{(t^2+\varepsilon)^{3/2}(\sqrt{1+\varepsilon} - \sqrt{\varepsilon})}\right| \le \frac{1}{\sqrt{\varepsilon}(\sqrt{1+\varepsilon} - \sqrt{\varepsilon})} < \frac{5}{4}\varepsilon^{-1/2}
\end{aligned}$$

$$\begin{aligned}
|t^2 f_\varepsilon''(t)| &= \left|\frac{-\varepsilon t^2}{(t^2+\varepsilon)^{3/2}(\sqrt{1+\varepsilon} - \sqrt{\varepsilon})}\right| \le \left(\frac{\varepsilon^{2/3} t^{4/3}}{t^2+\varepsilon}\right)^{3/2} \frac{1}{\sqrt{1+\varepsilon} - \sqrt{\varepsilon}} \\
&< \left(\frac{2^{2/3}\varepsilon^{1/3}}{3}\right)^{3/2} \frac{5}{4} < \frac{1}{2}\varepsilon^{1/2}.
\end{aligned}$$

In the last inequality we used that

$$\varepsilon^{2/3} t^{4/3} = 2^{2/3}\varepsilon^{1/3} \sqrt[3]{\varepsilon (t^2/2)(t^2/2)} \le 2^{2/3}\varepsilon^{1/3} \frac{\varepsilon + t^2/2 + t^2/2}{3} = \frac{2^{2/3}\varepsilon^{1/3}}{3}(t^2+\varepsilon).$$

Then, we will make ε small and then η small in such a way that $\mathcal{L}\varphi \ge c(\beta)$. To make the computations easier, we will use the Pucci operator $\mathcal{M}^-$, and we will denote $t = \eta|x'|/x_n$. On the one hand, we can check that

$$\begin{aligned}
\frac{\partial^2 \varphi}{\partial x_n^2} &= x_n^{\beta-2}((\beta^2 - \beta) f_\varepsilon(t) + (2 - 2\beta) t f_\varepsilon'(t) + t^2 f_\varepsilon''(t)) \\
&> x_n^{\beta-2}\left((\beta^2 - \beta)\left(1 - \frac{5}{4}t\right) + (\beta - 1)\frac{40}{21}t^2 - \frac{1}{2}\varepsilon^{1/2}\right) \\
&> x_n^{\beta-2}\left((\beta - 1)\left(\beta - \frac{5\beta}{4}t + \frac{40}{21}t^2\right) - \frac{1}{2}\varepsilon^{1/2}\right).
\end{aligned}$$

Now, we compute the discriminant of the second order polynomial that we found:

$$\text{Discriminant}\left(\beta - \frac{5\beta}{4}t + \frac{40}{21}t^2\right) = \frac{25\beta^2}{16} - \frac{160\beta}{21} = \beta\left(\frac{25\beta}{16} - \frac{160}{21}\right) < 0.$$

Hence, the second order polyonmial is always positive and attains a minimum $m_\beta > 0$. Choose ε such that $\varepsilon^{1/2} < (\beta-1)m_\beta$. Then,

$$\frac{\partial^2\varphi}{\partial x_n^2} > x_n^{\beta-2}\left((\beta-1)m_\beta - \frac{1}{2}\varepsilon^{1/2}\right) > x_n^{\beta-2}\frac{(\beta-1)m_\beta}{2} =: c_\beta x_n^{\beta-2} > 0$$

Consider now $i = 1, \dots, n-1$.

$$\begin{aligned}\left|\frac{\partial^2\varphi}{\partial x_i^2}\right| &= x_n^{\beta-2}\left|\eta^2 t^{-1} f_\varepsilon'(t)\frac{|x'|^2 - x_i^2}{|x'|^2} + \eta^2 f_\varepsilon''(t)\frac{x_i^2}{|x'|^2}\right| \\ &\le x_n^{\beta-2}(\eta^2|t^{-1}f_\varepsilon'(t)| + \eta^2|f_\varepsilon''(t)|) \\ &< x_n^{\beta-2}\eta^2\left(\frac{5}{4}\varepsilon^{-1/2} + \frac{5}{4}\varepsilon^{-1/2}\right) < x_n^{\beta-2}\eta^2\frac{5\varepsilon^{-1/2}}{2}.\end{aligned}$$

Now we need to compute the crossed derivatives. We begin with

$$\begin{aligned}\left|\frac{\partial^2\varphi}{\partial x_i\partial x_n}\right| &= x_n^{\beta-2}\left|\eta(\beta-1)\frac{x_i}{|x'|}f_\varepsilon'(t) - \eta^2\frac{x_i}{|x'|}f_\varepsilon''(t)\right| \\ &\le x_n^{\beta-2}(\eta(\beta-1)|f_\varepsilon'(t)| + \eta^2|f_\varepsilon''(t)|) \\ &< x_n^{\beta-2}\left(\eta\frac{5(\beta-1)}{4} + \eta^2\frac{5}{4}\varepsilon^{-1/2}\right) < x_n^{\beta-2}(\eta+\eta^2)\frac{5\varepsilon^{-1/2}}{2}.\end{aligned}$$

And finally, taking $i \neq j$ in $\{1, \dots, n-1\}$,

$$\begin{aligned}\left|\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right| &= x_n^{\beta-2}\left|-\eta^2 t^{-1} f_\varepsilon'(t)\frac{x_i x_j}{|x'|^2} + \eta^2 f_\varepsilon''(t)\frac{x_i x_j}{|x'|^2}\right| \\ &\le x_n^{\beta-2}(\eta^2|t^{-1}f_\varepsilon'(t)| + \eta^2|f_\varepsilon''(t)|) \\ &< x_n^{\beta-2}\eta^2\left(\frac{5}{4}\varepsilon^{-1/2} + \frac{5}{4}\varepsilon^{-1/2}\right) < x_n^{\beta-2}\eta^2\frac{5\varepsilon^{-1/2}}{2}.\end{aligned}$$

Define $H(x) = D^2\varphi(x)$, and also $H_0(x)$ to be the matrix with $\partial^2\varphi/\partial x_n^2$ at the lower right corner and zeros in all other entries. On the one hand, by the definition of $\mathcal{M}^-$:

$$\mathcal{M}^-(H_0) \ge \lambda x_n^{\beta-2} c_\beta.$$

Moreover, using that $\|H - H_0\|$ is bounded by the sum of the coefficients,

$$\mathcal{M}^-(H) \ge \mathcal{M}^-(H_0) - \Lambda\sum_{i,j=1}^{n}|(H-H_0)_{ij}| \ge x_n^{\beta-2}F(\eta),$$

where

$$F(\eta) = \lambda c_\beta - 5\Lambda(n-1)(\eta+\eta^2)\varepsilon^{-1/2} - \frac{5\Lambda(n-1)^2}{2}\eta^2\varepsilon^{-1/2}.$$

Since $\varepsilon > 0$ is fixed, we choose η small enough such that $F(\eta) \ge \lambda c_\beta/2$. To end the proof,

$$\mathcal{M}^-(D^2\varphi) = \mathcal{M}^-(H) \ge x_n^{\beta-2}\frac{\lambda c_\beta}{2} \ge \frac{\lambda c_\beta}{2} =: c(\beta),$$

where we use that $x_n \le 1$ and $\beta - 2 < 0$.

By the comparison principle, we conclude that $u(te_n) \ge \varphi(te_n) = t^\beta$. □

Remark 4.3.2. The constant L_0 in Theorem 4.1.2 is limited, in fact, by the value of η from this lemma, because the domain must contain wide enough cones, so the Lipschitz constant of the boundary must be small enough.

To prove the nondegeneracy property for solutions of divergence form equations, we proceed by approximation. The continuity assumption on the coefficients in (4.3) is necessary, see Proposition 4.6.3.

The following lemma is a natural approximation property of divergence form equations.

Lemma 4.3.3. *Let Ω be a bounded Lipschitz domain and $K \subset \Omega$ a compact subset. Let $\mathcal{L}_1, \mathcal{L}_2$ be divergence form operators, and let $u_1, u_2 \in H^1(\Omega)$ be the solutions of the following Dirichlet problems*

$$\begin{cases} \mathcal{L}_1 u_1 &= 0 \quad \text{in } \Omega \\ \quad u_1 &= g \quad \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}_2 u_2 &= 0 \quad \text{in } \Omega \\ \quad u_2 &= g \quad \text{on } \partial\Omega, \end{cases}$$

with $g \in H^1(\Omega)$ and

$$\mathcal{L}_1 u_1 = \mathrm{Div}(A_1(x)\nabla u_1), \quad \mathcal{L}_2 u_2 = \mathrm{Div}(A_2(x)\nabla u_2).$$

Then,

$$\|u_1 - u_2\|_{L^\infty(K)} \le C\{\|A_1 - A_2\|_{L^\infty(\Omega)}, \|A_1 - A_2\|^\tau_{L^\infty(\Omega)}\},$$

where $C > 0$ and $\tau \in (0,1)$ depend only on K, Ω, g and the ellipticity constants.

Proof. Since $u_1 = u_2$ on $\partial\Omega$, we can use $v = u_1 - u_2$ as a test function in $H^1_0(\Omega)$, to obtain

$$\int_\Omega \nabla u_1^\top A_1 \nabla v = \int_\Omega \nabla u_2^\top A_2 \nabla v = 0,$$

so

$$0 = \int_\Omega (\nabla u_1^\top A_1 - \nabla u_2^\top A_2)\nabla v = \int_\Omega \nabla v^\top A_1 \nabla v + \nabla u_2^\top (A_1 - A_2)\nabla v$$

and thus

$$\begin{aligned} \lambda \|\nabla v\|^2_{L^2(\Omega)} &\le \int_\Omega \nabla v^\top A_1 \nabla v = -\int_\Omega \nabla u_2^\top (A_1 - A_2)\nabla v \\ &\le \|A_1 - A_2\|_{L^\infty(\Omega)} \|\nabla u_2\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}. \end{aligned}$$

Hence, using that the H^1 norm of u_2 can be bounded by a constant depending on the domain, the ellipticity constants and the boundary data,

$$\|\nabla v\|_{L^2(\Omega)} \le C_1 \|A_1 - A_2\|_{L^\infty(\Omega)}.$$

This, combined with the Poincaré inequality, yields $\|v\|_{L^2(\Omega)} \le C_2 \|A_1 - A_2\|_{L^\infty(\Omega)}$.

On the other hand, let $\delta = d(K, \partial\Omega)$ and define the enlarged compact set $K' = \{x \in \Omega : d(x, K) \le \delta/2\}$. By the De Giorgi-Nash-Moser theorem, we have $\|u_i\|_{C^{0,\alpha}(K')} \le C_3$, where α and C_3 depend only on the domain, the dimension and the ellipticity constants, thus $\|v\|_{C^{0,\alpha}(K')} \le 2C_3$.

Let $p \in K$ such that $|v|$ reaches its maximum, and assume without loss of generality that $v(p) > 0$. Then, for all $x \in B_{\delta/2}$, $v(p+x) \ge v(p) - 2C_3|x|^\alpha$, and we can estimate $\|v\|_{L^2(\Omega)}$. The first observation is that $v(p+x) \ge v(p)/2$ when x is small enough, quantitatively,

$$v(p+x) \ge v(p)/2 \quad \iff \quad 2C_3|x|^\alpha \le v(p)/2 \quad \iff |x| \le C_4 v(p)^{1/\alpha},$$

so now we can use $v(p+x) \geq v(p)\chi_E/2$, $E = B_{C_4 v(p)^{1/\alpha}}$ to obtain

$$\|v\|_{L^2(\Omega)} \geq \left(\int_{B_{\delta/2}(p)} v^2\right)^{1/2} = \left(\int_{B_{\delta/2}} v(p+x)^2\right)^{1/2}$$
$$\geq \left(\int_{B_{\delta/2}} v(p)^2\chi_E/4\right)^{1/2} \geq \min\{|B_{\delta/2}|, |E|\}^{1/2} v(p)/2.$$

This presents us with two cases. When $B_{\delta/2} \subset E$, $v(p) \leq C_5\|v\|_{L^2(\Omega)}$. On the other hand, if $E \subset B_{\delta/2}$, $v(p)^{1+1/\alpha} \leq C_6\|v\|_{L^2(\Omega)}$. In either case,

$$v(p) \leq C_7 \max\{\|v\|_{L^2(\Omega)}, \|v\|_{L^2(\Omega)}^{\frac{\alpha}{\alpha+1}}\} \leq C \max\{\|A_1 - A_2\|_{L^\infty(\Omega)}, \|A_1 - A_2\|_{L^\infty(\Omega)}^{\frac{\alpha}{\alpha+1}}\},$$

and the result follows. □

As a consequence, we can prove the analogue of Lemma 4.3.1 for divergence form equations.

Lemma 4.3.4. *Let $\mathcal{L}$ be in divergence form with continuous coefficients, with modulus of continuity σ as in (4.3). Let $\beta' > 1$. There exists sufficiently small $\eta' > 0$ such that the following holds.*

Let u be a solution of

$$\begin{cases} \mathcal{L}u & \leq & 0 & \text{in } C_{2,\eta'} \\ u & \geq & 1 & \text{in } \{x_n > 1\} \cap C_{2,\eta'} \\ u & \geq & 0 & \text{on } \partial C_{2,\eta'}. \end{cases}$$

Then,

$$u(te_n) \geq t^{\beta'}, \quad \forall t \in (0, t_\sigma),$$

where

$$C_{2,\eta'} := \{x \in \mathbb{R}^n : \eta'|x'| < x_n < 2\}.$$

The constants t_σ and η' are positive and depend only on the dimension, β', σ, λ and Λ.

Proof. We will assume without loss of generality that $\beta' \in (1,2)$ and that $\mathcal{L}u = 0$ in $C_{2,\eta'}$. Let β, γ such that $\beta' > \gamma > \beta > 1$. Let $\eta > 0$ be the one provided by Lemma 4.3.1 with exponent β. Let $\eta' < \eta/8$ and $k_0 \in \mathbb{Z}^+$, to be chosen later. We will prove by induction that $u(2^{-k}e_n) \geq c2^{-k\gamma}$ for all integer $k \geq k_0$ and some $c > 0$. Notice that this implies that $u(te_n) \geq c't^\gamma$ for some smaller $c' > 0$ by a direct application of interior Harnack. To end the proof, notice that if t is small enough, since $\beta' > \gamma$,

$$u(te_n) \geq c't^\gamma \geq t^{\beta'}.$$

We proceed now with the induction. First, we define the following auxiliary functions.

$$b(x) = \frac{x_n}{2^{-k}}\tilde{b}\left(\frac{|x'|}{x_n}\right), \quad \tilde{b}(t) = \begin{cases} 1 & t < B, \\ 4 - 3t/B & B \leq t < 4B/3, \\ 0 & \text{otherwise,} \end{cases}$$

with $B = 3/(2\eta)$. We also write $b_1(x', x_n) = \tilde{b}(|x'|/x_n)$ for convenience of the notation.

We claim that there exists $c > 0$ such that, for all integer $k \geq k_0$, $u \geq c2^{-k\gamma}b_1$ in the $(n-1)$-dimensional ball $B'_{2^{2-k}/\eta} \times \{2^{-k}\}$.

For the first k_0, first observe that $u \geq 0$ everywhere by the maximum principle. Then, apply the interior Harnack inequality to the cylinder $B'_{2^{2-k}/\eta} \times [2^{-k}, 3/2]$, which is compactly contained in $C_{2,\eta'}$. Since $\sup u = 1$ in the cylinder, we have $u \geq m > 0$, and using that $b_1 \leq 1$, $u \geq m b_1$ in $B'_{2^{2-k}/\eta} \times \{2^{-k}\}$ and we can choose c accordingly.

Now, for the inductive step, let $K = B'_{2^{1-k}/\eta} \times \{2^{-k-1}\}$, which is compactly contained in $C_{2^{-k},\eta'}$, and let v and v_0 the solutions of the following Dirichlet problems

$$\begin{cases} \mathcal{L}v &= 0 & \text{in } C_{2^{-k},\eta'} \\ v &= 2^{-k\gamma} c b(x) & \text{on } \partial C_{2^{-k},\eta'}, \end{cases} \quad \textit{and} \quad \begin{cases} \mathcal{L}_0 v_0 &= 0 & \text{in } C_{2^{-k},\eta'} \\ v_0 &= 2^{-k\gamma} c b(x) & \text{on } \partial C_{2^{-k},\eta'}, \end{cases}$$

with $\mathcal{L}_0 v_0 := \mathrm{Div}(A_0 \nabla v_0)$, $A_0 = A(0)$.

Observe that $v = v_0 = 0$ on the *lateral* boundary of the cone $C_{2^{-k},\eta'}$. Then, it is clear that $u \geq v$ from the boundary conditions. Furthermore, by a rescaling of Lemma 4.3.3,

$$\|v - v_0\|_{L^\infty(K)} \leq 2^{-k\gamma} c C \max\{\|A - A_0\|_{L^\infty(C_{2^{-k},\eta'})}, \|A - A_0\|^\tau_{L^\infty(C_{2^{-k},\eta'})}\}.$$

For each $p \in K$, consider the cone $\mathcal{C}'$ with vertex in $(p', \eta'|p'|) \in \partial C_{2,\eta'}$ and slope η,

$$\mathcal{C}' := \{(x', x_n) \in \mathbb{R}^n : \eta|x' - p'| + \eta'|p'| < x_n < 2^{-k}\}.$$

Since $\eta' > \eta$, $\mathcal{C}' \subset C_{2,\eta'}$. Hence, $u \geq 0$ in $\partial\mathcal{C}'$. Moreover, by construction, the top part, $\{x_n = 2^{-k}\} \cap \overline{\mathcal{C}'}$ is contained in $B'_{2^{2-k}/\eta} \times \{2^{-k}\}$. Hence, we can apply a rescaled Lemma 4.3.1 to the normalized $2^{k\gamma} v_0$, because $\mathcal{L}_0$ has constant coefficients and is also a non-divergence form operator to obtain

$$2^{k\gamma} v_0(p) \geq c \left(\frac{2^{-k-1} - \eta'|p'|}{2^{-k} - \eta'|p'|} \right)^\beta \geq c \left(\frac{2^{-k-1} - 2^{2-k}\eta'/\eta}{2^{-k} - 2^{2-k}\eta'/\eta} \right)^\beta = c \left(\frac{\eta - 8\eta'}{2\eta - 8\eta'} \right)^\beta.$$

Then, passing the information on v_0 to v,

$$\begin{aligned} v(p) &\geq c \left(\frac{\eta - 8\eta'}{2\eta - 8\eta'} \right)^\beta - \|v - v_0\|_{L^\infty(K)} \\ &\geq 2^{-k\gamma} c \left(\left(\frac{\eta - 8\eta'}{2\eta - 8\eta'} \right)^\beta - C \max\{\|A - A_0\|_{L^\infty(C_{2^{-k},\eta'})}, \|A - A_0\|^\tau_{L^\infty(C_{2^{-k},\eta'})}\} \right) \\ &\geq 2^{-k\gamma} c (1/2)^\gamma, \end{aligned}$$

where for the last inequality we first choose a small η' such that

$$\left(\frac{\eta - 8\eta'}{2\eta - 8\eta'} \right)^\beta \geq \left(\frac{1}{2} \right)^\gamma + \delta,$$

with $\delta > 0$, and then take k large and use that A is continuous. Hence, if η' is small enough and k_0 is large enough in the first place, the inductive step holds. □

Now we are ready to prove Theorem 4.1.2. We divide the proof into three parts: an upper bound, a lower bound, and the proof of the $\mathcal{C}^{0,\alpha}$ regularity of the quotient.

4.3.2 Upper bound

We follow the arguments of [73]; see also [170]. The first lemma is a geometric fact that will make subsequent computations easier.

Lemma 4.3.5. *Let Ω be a Lipschitz domain as in Definition 4.1.1, with Lipschitz constant $L < 1/16$. Let*

$$A = \{x \in \Omega \cap \overline{B_{1-\delta}} : d(x, \partial\Omega) \geq \delta\} = \{x \in \Omega \cap B_1 : d(x, \partial(\Omega \cap B_1)) \geq \delta\},$$

where $\delta \in (0, 1/3)$. Then A is star-shaped with respect to the point $e_n/2$.

Proof. It is easy to check that $d(A, \partial(\Omega \cap B_1)) = \delta$, and that $e_n/2 \in A$ (since $L < 1/16$, $d(e_n/2, \partial\Omega) \geq 1/(2\sqrt{L^2+1}) > 1/3$).

We distinguish the *upper* and the *lower* boundaries of A as:

$$\partial_u A = \{x \in \partial A : d(x, \partial\Omega) > \delta\}, \quad \partial_l A = \{x \in \partial A : d(x, \partial\Omega) = \delta\}.$$

The first step is proving that $\partial_l A$ is a Lipschitz graph with the same or lower Lipschitz constant. For this, consider the set $\Omega_\delta = \{x \in B'_{1-\delta} \times \mathbb{R} : d(x, \partial\Omega) > \delta\}$, which contains the points above $\partial_l A$. For every vertical line l passing through $(x', 0)$, with $x' \in B'_{1-\delta}$, the set $l \cap \Omega_\delta$ is not empty, so we can define $h : B'_{1-\delta} \to \mathbb{R}$ as

$$h(x') = \inf\{x_n : d((x', x_n), \partial\Omega) > \delta\}.$$

Then, for a given $x' \in B'_{1-\delta}$, $(x', y) \in \Omega_\delta$ for all $y > h(x')$. Indeed, for every point $z = (z', z_n)$ in $\partial\Omega$, either $|z' - x'| > \delta$, and hence $d((x', y), z) > \delta$, or $|z' - x'| \leq \delta$. In this case, $z_n = g(z') \leq g(x') + L|z' - x'| < g(x') + \delta/16 \leq h(x') - \delta + \delta/16 < h(x')$, and then, $d((x', y), z) > d((x', x_n), z) = \delta$, because $y > h(x') > z_n$.

In any case, we have proven that $\Omega_\delta = \{(x', x_n) \in B'_{1-\delta} \times \mathbb{R} : x_n > h(x')\}$. Moreover, this shows that $\partial_l A$ is a subset of the graph of h. Now we want to see that h is Lipschitz. Notice that we can also define h with the complement set,

$$h(x') = \sup\{x_n : d((x', x_n), \partial\Omega) \leq \delta\} = \sup\{x_n : d((x', x_n), \partial\Omega) = \delta\}.$$

This can be seen as the superior envolvent of a union of spheres of radius δ centered at every point of $\partial\Omega$, hence

$$h(x') = \sup\{g(x' + t) + \sqrt{\delta^2 - |t|^2}, t \in B'_\delta\}.$$

Since this is a supremum of equi-Lipschitz functions, h is also Lipschitz with the same or lower constant, $L' \leq L < 1/16$. From $g(0) = 0$ we can also derive $h(0) \geq \delta$, and $h(0) \leq \delta\sqrt{L^2+1} < 1.02\delta$.

Now we will see that A is star-shaped with *center* at $e_n/2$, constructing a segment from $e_n/2$ to every point in A that lies entirely inside A. Let $p \neq e_n/2$ be a point in A, and let $q = (q', q_n)$ be the intersection of the line through p and $e_n/2$ and ∂A, that lies on the side of p and is furthest from $e_n/2$. We will see later that there is only one intersection at each side, but considering the furthest is enough for now.

If q lies in $\partial_l A$, $q_n = h(q')$. If q lies in $\partial_u A$, the point is above $\partial_l A$ and $q_n > h(q')$. In any case, we have always $q_n \geq h(q')$. It is clear that the segment $\overline{(e_n/2)q}$, that can be parametrised by $\{(tq', (1-t)/2 + tq_n), t \in [0,1]\}$ is contained in $\overline{B_{1-\delta}}$. We will prove that it lies entirely *above* $\partial_l A$ (except maybe in the point q), so it has not other intersections with ∂A besides q. We distinguish two cases:

If $q_n \geq 7/16$, for any point tq' inside the segment joining 0 and q' in $B'_{1-\delta}$ (this means $t \in (0,1)$), using that h is Lipschitz,

$$h(tq') \leq h(0) + L|tq'| < 1.02\delta + t/16 < 0.34 + t/16.$$

Moreover, the height of the segment $\overline{(e_n/2)q}$ above the point tq' is

$$(1-t)/2 + tq_n \geq 0.5 + (q_n - 0.5)t \geq 0.5 + (h(q') - 0.5)t \geq 0.5 - t/16,$$

and $0.5 - t/16 > 0.34 + t/16$ because $t/8 < 1/8 < 0.5 - 0.34 = 0.16$. Combining the two inequalities, $h(tq') < (1-t)/2 + tq_n$ as required.

On the other hand, if $q_n < 7/16$, $h(q') < 7/16$ as well. Since h is Lipschitz,

$$h(tq') \leq h(q') + L|q' - tq'| < q_n + (1-t)/16 = (q_n + 1/16) - (1/16)t.$$

The height of the segment $\overline{(e_n/2)q}$ above the point tq' is

$$(1-t)/2 + tq_n = 1/2 - (1/2 - q_n)t,$$

and $1/2 - (1/2 - q_n)t > (q_n + 1/16) - (1/16)t$ for $t \in (0,1)$ by a simple calculation.

Hence, in any case the segment joining $e_n/2$ and q crosses ∂A at q for the first time, implying A is star-shaped. $\square$

Now, we derive an interior Harnack inequality for domains with the shape we want to consider.

Lemma 4.3.6. *Let Ω be a Lipschitz domain as in Definition 4.1.1, with Lipschitz constant $L < 1/16$. Let $\delta \in (0, 1/3)$. Let $\mathcal{L}$ be as in (4.2) or (4.3). Let u be a positive solution, in the L^n-viscosity or the weak sense, of*

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \cap B_1 \\ u = 0 & \text{on } \partial\Omega \cap B_1, \end{cases}$$

with $f \in L^n(B_1)$. Let $A = \{x \in \Omega \cap \overline{B_{1-\delta}} : d(x, \partial\Omega) \geq \delta\}$. Then,

$$\sup_A u \leq C(\inf_A u + \|f\|_{L^n(B_1)}),$$

with C depending on the dimension, δ, λ and Λ, but not on the particular shape of Ω.

Proof. Let $x \in A$, and we will denote $y = e_n/2$ to simplify the notation. Since $A \subset B_{1-\delta}$, $|x - y| < 2$. We define

$$m := \left\lceil \frac{8}{\delta} \right\rceil.$$

Take $x_0 = x, \ldots, x_m = y$ a uniform partition on the segment $\overline{xy}$. It is clear that $|x_{i+1} - x_i| < \delta/4$. Then, consider the balls $B_\delta(x_i)$. We apply the interior Harnack inequality to obtain that

$$\sup_{B_{\delta/2}(x_i)} u \leq C\left(\inf_{B_{\delta/2}(x_i)} u + \delta\|f\|_{L^n(B_\delta(x_i))}\right).$$

In particular, $u(x_i) \le C(u(x_{i+1}) + \delta\|f\|_{L^n(B_i)}) \le C(u(x_{i+1}) + \delta\|f\|_{L^n(B_1)})$, and iterating this, $u(y) \le C^{m+1}u(x) + C'\|f\|_{L^n(B_1)}$. Taking the points in reverse order yields $u(x) \le C^{m+1}u(y) + C'\|f\|_{L^n(B_1)}$.

Now take $x, z \in A$, and apply the inequalities between $u(x)$ and $u(y)$ to $u(y)$ and $u(z)$. We can put them together finally to get

$$u(x) \le C^{2(m+1)}u(z) + C''\|f\|_{L^n(B_1)}, \quad u(z) \le C^{2(m+1)}u(x) + C''\|f\|_{L^n(B_1)}.$$

Finally, notice that C, m and C'' do not depend on the shape of Ω. □

The next step is the following lemma, that shows that the condition $u > 0$ and $u(e_n/2) \le 1$ implies $\|u\|_{L^p(B_1)} \le c_p$ in Theorem 4.1.2.

Lemma 4.3.7. *Let Ω a Lipschitz domain as in Definition 4.1.1, with Lipschitz constant $L < 1/16$. Let $\mathcal{L}$ be as in (4.2) or (4.3). Let u be a positive solution, in the L^n-viscosity or the weak sense, of*

$$\begin{cases} \mathcal{L}u &= f \quad \text{in } \Omega \cap B_1 \\ u &= 0 \quad \text{on } \partial\Omega \cap B_1, \end{cases}$$

such that $u(e_n/2) \le 1$, with $f \in L^n(B_1)$. Then, there exist $p, C_p > 0$ such that

$$\|u\|_{L^p(B_1)} \le C_p,$$

with p and C_p only depending on the dimension, λ, Λ and $\|f\|_{L^n(B_1)}$.

Proof. We will prove that there exist a sequence $\{a_k\}$ and some positive c and b such that $\sup u \le a_k \le cb^k$ in the sets $A_k = \{x \in \Omega \cap B_{1-2^{-k}} : d(x, \partial\Omega) > 2^{-k}\}$ for all $k \ge 3$. This means roughly that $\sup u$ grows at most like d^{-K} for some big K, and then u^p will be integrable if $p > 0$ is small enough.

First, by Lemma 4.3.6 applied with $\delta = 1/8$, together with the fact that $u \le 1$ in at least a point of A_3, $\sup_{A_3} \le C(1 + \|f\|_{L^n(B_1)}) =: a_3$.

Now, we will show that $a_{k+1} \le c_1 a_k + c_2$, with $c_i > 0$. This easily implies by induction that $a_k \le cb^k$ for some $b, c > 0$.

Take $x \in A_{k+1}$. We will prove that there exists a close $y \in A_k$ such that $d(x, y) < 2^{-k+3}$. In fact, let $z = (z', z_n)$ be the intersection of ∂A_k with the segment $\overline{(e_n/2)x}$. Proving $d(x, z) < 2^{-k+3}$ will suffice, because there are points in A_k arbitrarily close to z.

We can parametrise the segment as $\psi(t) = t(e_n/2) + (1-t)x$, with $t \in (0, 1)$. Then, $z = \psi(t_*)$, where we define

$$t_* := \inf I_k = \inf\{t \in (0, 1) : \psi(t) \in A_k\}.$$

Since A_k is star-shaped with rays coming from $e_n/2$ by Lemma 4.3.5, and it contains an open ball around $e_n/2$, I_k is an open interval. Looking closely at the definition of A_k, we can write I_k as the intersection of two conditions:

$$I_k = (t_1, 1) \cap (t_2, 1) := \{t \in (0, 1) : \psi(t) \in B_{1-2^{-k}}\} \cap \{t \in (0, 1) : d(\psi(t), \partial\Omega) > 2^{-k}\}.$$

First, the condition $\psi(t) \in B_{1-2^{-k}}$ means $|te_n/2 + (1-t)x| < 1 - 2^{-k}$, which is automatically fulfilled when $t \ge 2^{-k+1}$, because then

$$|te_n/2 + (1-t)x| \le t/2 + (1-t)|x| < t/2 + (1-t) = 1 - t/2.$$

Hence $t_1 \leq 2^{-k+1}$. To finish this argument we need an upper bound on t_2 as well. Take an arbitrary $t \in [2^{-k+2}, 1]$, and we will see that $d(\psi(t), \partial\Omega) > 2^{-k}$. To do so, we will prove that $\psi_n(t) > g(\psi'(t)) + 2^{-k}\sqrt{L^2+1}$, with $\psi(t) = (\psi'(t), \psi_n(t))$ as usual. Since g is Lipschitz with constant $L < 1/16$, $|g(x')| < 1/16$, and

$$g(\psi'(t)) \leq g(x') + L|x' - \psi'(t)| < g(x') + t/16,$$

we deduce

$$\begin{aligned}\psi_n(t) = t/2 + (1-t)x_n > t/2 + (1-t)g(x') = g(x') + t(1/2 - g(x'))\\ > g(x') + 7t/16,\end{aligned}$$

and

$$g(\psi'(t)) + 2^{-k}\sqrt{L^2+1} < g(x') + t/16 + 2^{-k}\sqrt{L^2+1} < g(x') + t/16 + 3 \cdot 2^{-k-1}.$$

Finally, since $6t/16 \geq 3 \cdot 2^{-k-1}$, $\psi_n(t) > g(\psi'(t)) + 2^{-k}\sqrt{L^2+1}$ as desired and $t_2 \leq 2^{-k+2}$. Now, $t_* = \max\{t_1, t_2\} \leq 2^{-k+2}$, and this implies $d(x, z) = t_* d(x, e_n/2) < 2t_* \leq 2^{-k+3}$.

Now, for a given $x \in A_{k+1}$, we have $y \in A_k$ such that $d(x, y) < 2^{-k+3}$. Consider a uniform partition in 32 pieces of the segment $\overline{xy}$, $p_0 = y, \ldots, p_{31} = x$. Since A_{k+1} is star-shaped, $\overline{xy} \subset A_{k+1}$, so the balls $B_i = B_{2^{-k-1}}(p_i)$ are completely contained in $\Omega \cap B_1$. Now, $d(p_i, p_{i+1}) < 2^{-k-2}$, and applying the interior Harnack inequality we get $u(p_{i+1}) \leq C(u(p_i) + \|f\|_{L^n(B_i)}) \leq C(u(p_i) + \|f\|_{L^n(B_1)})$. Iterating this inequality, $u(y) \leq c_1 u(x) + c_2$, for some constants c_1, c_2 only depending on the dimension and $\|f\|_{L^n(B_1)}$.

Now we know that $\sup u \leq cb^k$ in A_k. Let $p = \log_b \sqrt{2}$, and compute the L^p norm of u:

$$\begin{aligned}\int_{B_1} |u|^p &= \int_{A_3} |u|^p + \sum_{j=3}^{\infty} \int_{A_{j+1}\setminus A_j} |u|^p \leq |A_3| cb^{3p} + \sum_{j=4}^{\infty} |A_j \setminus A_{j-1}| c(b^p)^j\\ &\leq c\left(2\sqrt{2}|A_3| + \sum_{j=3}^{\infty} |A_{j+1} \setminus A_j| 2^{j/2}\right) \leq c\left(2\sqrt{2}|B_1| + \sum_{j=3}^{\infty} 2^{-j} V(n) 2^{j/2}\right)\\ &= c\left(2\sqrt{2}|B_1| + V(n)(1+\sqrt{2})/2\right) =: C_p^p,\end{aligned}$$

where we have used that $|A_{j+1} \setminus A_j| \leq 2^{-j} V(n)$. We will prove it now.

$$A_{j+1} \setminus A_j \subset (B_1 \setminus B_{1-2^{-j}}) \cup \{x \in B_1 : d(x, \partial\Omega) \leq 2^{-j}\}.$$

On the one hand, $|B_1 \setminus B_{1-2^{-j}}| \leq 2^{-j}|\partial B_1|$, where $|\partial B_1|$ is the $(n-1)$-dimensional measure of the boundary of the ball B_1. On the other hand, the second set is a subset of the *thickening* of $\partial\Omega$ in the e_n direction, with height $2^{-j}\sqrt{L^2+1}$ at each side:

$$\left\{(x', x_n) \in B_1' \times \mathbb{R} : |x_n - g(x')| \leq 2^{-j}\sqrt{L^2+1}\right\}.$$

The measure of this second set is $2^{-j+1}\sqrt{L^2+1}|B_1'|$ (again using the measure of $\mathbb{R}^{n-1}$). Hence, defining $V(n) = |\partial B_1| + 2\sqrt{1/16^2+1}|B_1'|$ serves our purpose. □

The previous Lemma 4.3.7 implies that $u \in L^p(B_1)$. Then, we can use Theorem 4.2.6 to obtain the following L^∞ bound on u:

Proposition 4.3.8. *Let Ω be a Lipschitz domain as in Definition 4.1.1, $\mathcal{L}$ as in (4.2) or (4.3) and $r \in (0,1)$. Let u be a L^n-viscosity or weak solution of*

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \cap B_1 \\ u = 0 & \text{on } \partial\Omega \cap B_1, \end{cases}$$

with $f \in L^n(\Omega)$.

Then, for all $p > 0$, if $u \in L^p(\Omega \cap B_1)$, u is bounded in B_r, i.e.

$$\sup_{B_r} u \leq K(\|u\|_{L^p(B_1)} + \|f\|_{L^n(\Omega)}),$$

with $K = K(n, p, \lambda, \Lambda, r)$.

Proof. Denote $v = u^+$, extending v by zero in $B_1 \setminus \Omega$, and extend f by zero in $B_1 \setminus \Omega$. Then, it is easy to check that $\mathcal{L}v \geq f$ in B_1. Now use Theorem 4.2.6 and a covering argument to get

$$\sup_{B_r} v \leq C_p(r)(\|u\|_{L^p(B_1)} + \|f\|_{L^n(B_1)}).$$

The conclusion trivially follows. □

4.3.3 Lower bound

The next step is to construct an iteration to see that solutions of $\mathcal{L}u = f$ that are sufficiently positive away from the boundary, and not very negative near it, are actually positive everywhere. As we have a right hand side $f \in L^q$, we need to be careful with the scaling, so we cannot use directly interior Harnack estimates to prove positivity, and we will need the nondegeneracy estimates in Lemmas 4.3.1 and 4.3.4.

Lemma 4.3.9. *Let $q > n$, $\kappa > 1$, and let $\mathcal{L}$ be as in (4.2) or (4.3). There exists $L_* = L_*(q, n, \kappa, \lambda, \Lambda)$ such that the following holds.*

Let Ω be a Lipschitz domain as in Definition 4.1.1 with constant $L < L_*$. Let f be such that $\|f\|_{L^q(B_1)} \leq c_0$. Let u be a L^n-viscosity or weak solution of

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \cap B_1 \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{with} \quad \begin{cases} u \geq 1 & \text{in } \Omega \cap \{x \in B_1 : d(x, \partial\Omega) > \delta\} \\ u \geq -\varepsilon & \text{in } \Omega \cap B_1. \end{cases} \tag{4.10}$$

Then,

$$\begin{cases} u \geq \rho^\kappa & \text{in } \Omega \cap \{x \in B_\rho : d(x, \partial\Omega) > \rho\delta\} \\ u \geq -\rho^\kappa \varepsilon & \text{in } \Omega \cap B_\rho \end{cases}$$

for some sufficiently small $\rho, \varepsilon, \delta, c_0 \in (0,1)$, with $\rho > 2\delta$, only depending on the dimension, κ, q, λ, Λ, as well as σ, when applicable.

Proof. Let $\beta \in (1, \kappa)$. Apply Lemma 4.3.1 in the non-divergence case (respectively, Lemma 4.3.4 in the divergence case) to obtain $\eta > 0$ (resp. η').

Let $h > 0$ to be chosen later. For $x_0 = (x_0', x_{0n})$ in $\{x \in B_\rho : d(x, \partial\Omega) > \rho\delta\}$, define the cone

$$\mathcal{C} = (x_0', g(x_0')) + hC_\eta = \{x \in \mathbb{R}^n : \eta|x' - x_0'| < x_n - g(x_0') < h\}.$$

Here, we distinguish the upper and the lateral boundaries, respectively,

$$\partial_u\mathcal{C} = \{x \in \mathbb{R}^n : \eta|x' - x_0'| \le x_n = g(x_0') + h\}$$
$$\partial_l\mathcal{C} = \{x \in \mathbb{R}^n : \eta|x' - x_0'| = x_n < h\}.$$

and the upper half cone

$$\mathcal{C}^+ = \mathcal{C} \cap \{x_n > g(x_0') + h/2\}.$$

Now, take $L_* = \min\{\eta/2, 1/16\}$. Hence, the slope of $\partial\Omega$ will be at most half of the slope of $\mathcal{C}$, so the cone separates from the boundary. By some geometric computations, we find $d(\partial_u\mathcal{C}, \partial\Omega) \ge h/\sqrt{4+\eta^2}$.

Let now $\rho \le 1/2$. The distance of the furthest points of $\mathcal{C}$ to $(x_0', g(x_0)')$ is $h\sqrt{1+1/\eta^2}$. Hence, taking $h \le 1/(2\sqrt{1+1/\eta^2})$ suffices to have $\mathcal{C} \subset \Omega \cap B_1$. Furthermore, making $h = 4\delta\sqrt{4+\eta^2}$, we will have $\partial_u\mathcal{C} \subset \{x \in B_1 : d(x, \partial\Omega) > \delta\}$, and also $\mathcal{C}^+ \subset \{x \in B_1 : d(x, \partial\Omega) > \delta\}$. Note that this forces δ to be small, but we will choose it at the end, so this is not a problem.

Define $\tilde{u}(x) = u(x_0', g(x_0') + hx) + \varepsilon$. Let $\tilde{u} = v + w$, where

$$\begin{cases} \mathcal{L}v = 0 & \text{in } C_\eta \\ v = \tilde{u} & \text{on } \partial C_\eta \end{cases} \quad \textit{and} \quad \begin{cases} \mathcal{L}w = f & \text{in } C_\eta \\ w = 0 & \text{on } \partial C_\eta. \end{cases}$$

By the ABP estimate, Theorem 4.2.2, in the non-divergence form case, or by Theorem 4.2.3 in the divergence form case, $\|w\|_{L^\infty(C_\eta)} \le C'\|f\|_{L^n(C_\eta)} \le C'c_0$. On the other hand, $v \ge 0$ on ∂C_η, and $v \ge 1$ on $\partial_u C_\eta$ and C_η^+ (defining the upper boundary and the upper half anologously). Hence, we can apply Lemma 4.3.1 or a rescaled Lemma 4.3.4 to v to conclude that $v(te_n) \ge t^\beta$, possibly only for small $t < t_\sigma$.

Putting all together, $\tilde{u}(te_n) \ge t^\beta - C'c_0$, which means

$$u((x_0', g(x_0')) + hte_n) \ge t^\beta - C'c_0 - \varepsilon,$$

only when $t < t_\sigma$ for divergence form operators. Therefore,

$$u(x_0) \ge \left(\frac{x_{0n} - g(x_0')}{h}\right)^\beta - C'c_0 - \varepsilon \ge \left(\frac{\rho\delta}{h}\right)^\beta - (C'c_0 + \varepsilon) = \left(\frac{\rho}{2\sqrt{4+\eta^2}}\right)^\beta - C'c_0 - \varepsilon.$$

Finally, since $\beta < \kappa$, we can choose $\rho > 0$ small enough such that $\rho^{\beta-\kappa} \ge 6\sqrt{4+\eta^2}$ (and $t < t_\sigma$ if needed), and then, choosing $\varepsilon, c_0 > 0$ small enough,

$$u(x_0) \ge 3\rho^\kappa - C'c_0 - \varepsilon \ge \rho^\kappa.$$

Now, for the second inequality, let $x_0 \in B_{1-3\delta}$, $d(x_0, \partial\Omega) \le \delta$. Let $v = u^-/\varepsilon$ in the ball $B_{3\delta(x_0)}$, extending u by 0 below $\partial\Omega$. By elementary properties of L^n-viscosity and weak solutions, since $\mathcal{L}u \le f$, $\mathcal{L}v \ge -f^+/\varepsilon$. Now, $v \ge 0$ in the whole ball, $v \le 1$ because $u \ge -\varepsilon$, and $v = 0$ below $\partial\Omega$. Let $z \in \partial\Omega$ be the closest point of the boundary to x_0. Let C_z be the downwards cone with slope L_* and vertex in z. Then, C_z lies entirely below $\partial\Omega$, and $v = 0$ in $C_z \cap B_{3\delta}(x_0)$. Since $d(x_0, z) \le \delta$, $|C_z \cap B_{3\delta}(x_0)| \ge c(L_*)|B_{3\delta}(x_0)|$, where $c(L_*)$ is a geometric constant that only depending on the dimension and L_*.

Applying Theorem 4.2.5, $v \le 1 - \gamma$ in $B_{3\delta/2}(x_0)$, and in particular $v(x_0) \le 1 - \gamma$. In order to do it, we need f^+/ε to be small enough, to have $\|f^+/\varepsilon\|_{L^n(B_1)} \le \delta(c_L)$ in the notation of the theorem.

We will iterate this reasoning with the functions $v_j = v/(1-\gamma)^j$, defined in $B_{3\delta}(x_0)$, with $x_0 \in B_{1-3j\delta}$, $d(x_0, \partial\Omega) \le \delta$. The conclusion of each iteration is $v_j \le 1-\gamma$ in $B_{1-3(j+1)\delta}$, i.e. $v_{j+1} \le 1$ in $B_{1-3(j+1)\delta}$. This implies $v \le (1-\gamma)^j$ in $B_{1-3j\delta}$, and then $u \ge -(1-\gamma)^j\varepsilon$ in $B_{1-3j\delta}$.

To end the proof we only need to choose j such that $(1-\gamma)^j < \rho^\kappa$, and then make δ small until $1-3j\delta \ge \rho$. Finally, notice that we need $\|f^+/((1-\gamma)^i\varepsilon)\|_{L^n(B_1)} \le \delta(c_L)$ for $i = 1, \dots, j$ to be able to apply successively Theorem 4.2.5. This is possible choosing c_0 accordingly once we know j. □

Now, we iterate the lemma to obtain the desired result.

Proposition 4.3.10. *Let $q > n$, $\kappa > 1$, and let $\mathcal{L}$ be as in (4.2) or (4.3). There exist $L_* = L_*(q, n, \kappa, \lambda, \Lambda) > 0$ and $\varepsilon, \delta, c_0 \in (0,1)$, such that the following holds.*

Let Ω be a Lipschitz domain as in Definition 4.1.1 with constant $L < L_*$. Let u be a solution of (4.10) with f such that $\|f\|_{L^q(B_1)} \le c_0$. Then,

$$u > 0 \quad \text{in } \Omega \cap B_{2/3}.$$

Moreover, for all $t \in (0,1)$,

$$u(te_n) \ge t^\kappa.$$

The constants L_*, ε, δ and c_0 depend only on the dimension, κ, q, λ, Λ, as well as σ, when applicable.

Proof. We will iterate the previous Lemma 4.3.9. Assume without loss of generality that $\kappa < 2 - n/q$. Let $u_0 = u$, $f_0 = f$, and define the scalings:

$$u_{j+1}(x) = \rho^{-\kappa} u_j(\rho x), \quad f_{j+1}(x) = \rho^{2-\kappa} f_j(\rho x).$$

Define Ω_j to be the rescaled domains of the u_j. Observe that the Lipschitz constant of the domains is the same or smaller, and that $\mathcal{L}u_j = f_j$. Now we will see that the right hand side is bounded as we need. Indeed, since $\rho^{2-\kappa} < \rho^{n/q}$,

$$\|f_{j+1}\|_{L^q(B_1)} < \left(\int_{B_1} \rho^n |f_j(\rho x)|^q \mathrm{d}x \right)^{1/q} = \left(\int_{B_\rho} |f_j(y)|^q \mathrm{d}y \right)^{1/q} = \|f_j\|_{L^q(B_\rho)},$$

and then $\|f_j\|_{L^q(B_1)} \le c_0$ for all j.

We will prove by induction that u_j satisfies (4.10) for all j as well. Start supposing u_j does. Then, by Lemma 4.3.9, $u_j \ge \rho^\kappa$ in $\Omega_j \cap \{x \in B_\rho : d(x, \partial\Omega_j) > \rho\delta\}$, which is equivalent to $u_{j+1} \ge 1$ in $\Omega_{j+1} \cap \{x \in B_\rho : d(x, \partial\Omega_{j+1}) > \rho\delta\}$. Also by the lemma, $u_j \ge -\rho^\kappa\varepsilon$ in B_ρ, which is the same as $u_{j+1} \ge -\varepsilon$ in B_1.

All iterates u_j satisfy (4.10), thus in particular $u_j(te_n) \ge 1$ for $t \in (2\delta, 1)$, taking into account that, since $L_* < \sqrt{3}$, $d(2\delta e_n, \partial\Omega) > \delta$. Rescaling back, this translates easily into $u(te_n) \ge t^\kappa$.

Now, observe that, after a change of variables, choosing smaller ε, δ and c_0 if needed, the function $\tilde{u}(x) = u(x_0 + x/3)$ is also a solution of (4.10) for any $x_0 \in \partial\Omega \cap B_{2/3}$. Analogously, we have $\tilde{u}(te_n) \ge t^\kappa$, thus $u(x_0 + te_n/3) > 0$, and since $\delta < 1/3$ this implies $u > 0$ in $\Omega \cap B_{2/3}$. □

As a consequence, we find:

Corollary 4.3.11. *Let $q > n$, $\kappa > 1$, and let $\mathcal{L}$ be as in (4.2) or (4.3). There exists $L_* = L_*(q, n, \kappa, \lambda, \Lambda)$ such that the following holds.*

Let Ω be a Lipschitz domain as in Definition 4.1.1 with constant $L < L_*$. Let f such that $\|f\|_{L^q(B_1)} \le c_0$. Let v be a positive L^n-viscosity or weak solution of $\mathcal{L}v = f$, with $v(e_n/2) \ge 1$. Then, for all $x \in B_{1/2}$,

$$v(x) \ge c_1 d(x, \partial\Omega)^\kappa,$$

for some sufficiently small $c_0, c_1 > 0$, only depending on the dimension, κ, q, λ, Λ, as well as σ, when applicable.

Remark 4.3.12. We can also write $x_n - g(x')$ instead of $d(x, \partial\Omega)$, since the two quantities are comparable.

Proof. Assume, after dividing by a constant if necessary, that $v(e_n/2) = 1$. Let $v'(x) = v(2x)$ and Ω' the corresponding scaled domain. By a version of Lemma 4.3.6, we have that if c_0 is small enough, $v' \ge c_2 > 0$ in $\{x \in \Omega' \cap B_{3/2} : d(x, \partial\Omega') > \delta\}$, for any $\delta > 0$, some small $c_2 > 0$ that depends only on δ, the dimension, q and the ellipticity constants. Now, apply the previous Proposition 4.3.10 to v'/c_2 in the balls $B_1(x_0)$ for any $x_0 \in B_1$ (we may need to ask that c_0 is smaller to do so). Hence, $v'(x_0 + te_n)/c_2 \ge t^\kappa$, and this implies $v(x_0 + (t/2)e_n) \ge c_2(t/2)^\kappa$. To end the proof, notice that $d(x_0 + (t/2)e_n, \partial\Omega) \in [t/2, t\sqrt{L^2+1}/2]$, so we can absorb the factor needed to change t for $d(x, \partial\Omega)$ in the constant c_1. □

4.3.4 Proof of the main result

Now we have all that we need to prove Theorem 4.1.2. Observe that Corollary 4.1.4 is a direct consequence. We divide the proof in two parts: in the first one we prove the inequality, and in the second we deduce the $\mathcal{C}^{0,\alpha}$ regularity of u/v.

Proof of Theorem 4.1.2. We prove the inequality first. Let

$$\kappa = 1 + \frac{1}{2}\left(1 - \frac{n}{q}\right) > 1,$$

and choose $L_0(q, n, \lambda, \Lambda) = L_*(q, n, \kappa, \lambda, \Lambda)$ with the definition of L_* given by Proposition 4.3.10. We will still keep κ explicit to simplify some calculations. If we are in the case $u > 0$, apply Lemma 4.3.7. In either case, by Proposition 4.3.8, $u \le K$ in $B_{3/4}$.

Then, consider v in the set $A = \{x \in \overline{B_{3/4}} : d(x, \partial\Omega) \ge 3\delta/4\}$. A is a subset of $\{x \in \overline{B_{1-3\delta/4}} : d(x, \partial\Omega) \ge 3\delta/4\}$. Hence, by Lemma 4.3.6, and from $v(e_n/2) \ge 1$, it follows that $v \ge C^{-1} - \|f\|_{L^n(B_1)} \ge C^{-1} - c_0$ in the whole set A. Furthermore, choosing $m = C^{-1}/2$ and $c_0 \le m$ yields $v \ge m > 0$ in A.

Define now

$$w := \frac{1+\varepsilon}{m} v - \frac{\varepsilon}{K} u,$$

with $\varepsilon > 0$ to be determined later. We will show that $w > 0$ in $B_{1/2}$, and therefore, taking $C = K(1+\varepsilon)/(m\varepsilon)$, $Cv - u > 0$.

By construction, $w \ge v/m \ge 1$ in A, and $w \ge -\varepsilon$ in $B_{3/4}$. To apply Proposition 4.3.10 (rescaled to the ball $B_{3/4}$), we need to estimate $\mathcal{L}w$:

$$\|\mathcal{L}w\|_{L^q(B_{3/4})} \le \frac{1+\varepsilon}{m}\|\mathcal{L}v\|_{L^q(B_1)} + \frac{\varepsilon}{K}\|\mathcal{L}u\|_{L^q(B_1)} \le \left(\frac{1+\varepsilon}{m} + \frac{\varepsilon}{K}\right) c_0.$$

Let $\tilde{w}(x) = w(3x/4)$. Then, $\tilde{w} \geq 1$ in $\Omega \cap \{x \in B_1 : d(x, \partial\Omega) \geq \delta\}$ and $\tilde{w} \geq -\varepsilon$ in $\Omega \cap B_1$. Choosing sufficiently small $\varepsilon, c_0 > 0$ to apply Proposition 4.3.10, we get $\tilde{w} > 0$ in $B_{2/3}$, thus $w > 0$ in $B_{1/2}$.

Now, for the boundary $\mathcal{C}^{0,\alpha}$ regularity of the quotient u/v, we will first prove the regularity for the boundary points, and then we will extend it to the whole closed domain $\overline{\Omega \cap B_{1/2}}$, where u/v is extended by continuity on $\partial\Omega$. These arguments are the standard ones found in the literature, but we have to be careful with some calculations to take into account the right hand side of the equations. Additionally, let c_0^* be the value for c_0 found in the first part of the proof. We will adjust the final value of c_0 in terms of this c_0^*.

By a covering argument, the inequality $u \leq C'v$ is valid in $\Omega \cap B_{3/4}$ with an appropriate constant C'. Since either $u > 0$ or we can interchange u by $-u$ and the hypotheses still hold, we have $u \geq -C'v$ as well. Let $x_0 \in \partial\Omega \cap \overline{B_{1/2}}$.

First, we will show by induction that there exist sequences $\{a_j\}, \{b_j\}$ such that, for every integer $j \geq 2$,

$$a_j v \leq u \leq b_j v \text{ in } \Omega \cap B_{2^{-j}}(x_0), \ (b_{j+1} - a_{j+1}) = (1-\theta)(b_j - a_j), \ \theta \in (0, 1 - 2^{1-\kappa}].$$

For $j = 2$ we take $a_j = -C', b_j = C'$, with the constant from the covering argument. Now, to perform the inductive step, we define two new functions:

$$w_1 := \frac{u - a_j v}{b_j - a_j}, \quad w_2 := \frac{b_j v - u}{b_j - a_j}.$$

These functions are positive solutions of $\mathcal{L}w_i = f_i$ in $\Omega \cap B_{2^{-j}}(x_0)$, vanish continuously at $\partial\Omega$, and $w_1 + w_2 = v$. Therefore, for one of them (the biggest in the point), $2w_i(x_0 + e_n/2^{j+1}) \geq v(x_0 + e_n/2^{j+1})$. To apply the boundary Harnack, we define the following rescaled functions, with $c_1 > 0$ from Corollary 4.3.11 in order to have $\tilde{v}(e_n/2) \geq 1$. Let

$$\tilde{v}(x) = c_1^{-1} 2^{j\kappa} v(x_0 + 2^{-j}x), \quad \tilde{w}_i(x) = c_1^{-1} 2^{j\kappa} w_i(x_0 + 2^{-j}x),$$
$$\tilde{f}_1(x) = 2^{j(\kappa-2)} \frac{f(x_0 + 2^{-j}x) - a_j g(x_0 + 2^{-j}x)}{c_1(b_j - a_j)},$$
$$\tilde{f}_2(x) = 2^{j(\kappa-2)} \frac{b_j g(x_0 + 2^{-j}x) - f(x_0 + 2^{-j}x)}{c_1(b_j - a_j)}.$$

Now we must check $\|\tilde{f}_i\|_{L^q(B_1)} \leq c_0^*$. Indeed, choosing c_0 appropriately,

$$\|\tilde{f}_1\|_{L^q(B_1)} \leq \frac{\|2^{j(\kappa-2)} f(x_0 + 2^{-j}x)\|_{L^q(B_1)} + a_j \|2^{j(\kappa-2)} g(x_0 + 2^{-j}x)\|_{L^q(B_1)}}{c_1(b_j - a_j)}$$
$$\leq c_0 \frac{2^{j(n/q+\kappa-2)}(1 + |a_j|)}{c_1(b_j - a_j)} \leq \frac{2^{j(n/q+\kappa-2)} c_0}{c_1(1-\theta)^{j-2}} \leq \frac{c_0(1-\theta)^2}{c_1} \leq c_0^*.$$

The same works for f_2.

Applying a rescaled version of the boundary Harnack inequality to the functions $2w_i, v$, we get that $w_i \geq \frac{v}{2C'}$ in $\Omega \cap B_{2^{-(j+1)}}(x_0)$. This presents two options: either

$$\frac{u - a_j v}{b_j - a_j} \geq \frac{v}{2C'} \quad \Rightarrow \quad u \geq \left(a_j + \frac{b_j - a_j}{2C'}\right) v =: \tilde{a}_{j+1} v, \quad \tilde{b}_{j+1} = b_j,$$

or

$$\frac{b_j v - u}{b_j - a_j} \geq \frac{v}{2C'} \quad \Rightarrow \quad u \leq \left(b_j - \frac{b_j - a_j}{2C'}\right) v =: \tilde{b}_{j+1} v, \quad \tilde{a}_{j+1} = a_j.$$

Either $\tilde{a}_{j+1} > a_j$ or $\tilde{b}_{j+1} < b_j$. We cannot choose them yet as a_{j+1}, b_{j+1}, because we need to ensure $1 - \theta \geq 2^{1-\kappa}$. This is done by choosing

$$\begin{aligned} a_{j+1} &= \min\{\tilde{a}_{j+1}, a_j + (1 - 2^{1-\kappa})(b_j - a_j)\}, \\ b_{j+1} &= \max\{\tilde{b}_{j+1}, b_j - (1 - 2^{1-\kappa})(b_j - a_j)\}. \end{aligned}$$

After this, $a_j \leq u/v \leq b_j$ in $\Omega \cap B_{2^{-j}}(x_0)$, and then

$$\sup_{B_{2^{-j}(x_0)\cap\Omega}} u/v - \inf_{B_{2^{-j}(x_0)\cap\Omega}} u/v \leq b_j - a_j = (2C')(1-\theta)^{j-2}. \tag{4.11}$$

We can extend u/v by continuity at x_0 as the limit of the a_j (or the b_j), and for any point $p \in \Omega \cap B_{2^{-j}}(x_0)$, $|(u/v)(x_0) - (u/v)(p)| \leq 2C'(1-\theta)^{j-2}$, hence u/v is $\mathcal{C}^{0,\alpha}$ at x_0 with $\alpha = -\log_2(1-\theta)$. Then, for every point x_0 on the boundary we have

$$\left|\frac{u}{v}(x_0) - \frac{u}{v}(p)\right| \leq C|x_0 - p|^\alpha,$$

for some uniform constant $C > 0$, for any $p \in B_{1/2} \cap \overline{\Omega}$.

Now, for the interior points, let $x_1, x_2 \in B_{1/2}$, $d = |x_1 - x_2|$ and $\delta_i = d(x_i, \partial\Omega)$. There are three different cases:

Case 1. If $d \geq 1/16$, we just use the fact that $-C'v \leq u \leq C'v$ in $\Omega \cap B_{3/4}$, hence, for any $\alpha \in (0,1)$,

$$\left|\frac{u}{v}(x_1) - \frac{u}{v}(x_2)\right| \leq 2C' \leq 32C'|x_1 - x_2|^\alpha.$$

Case 2. If the points are far compared with the distance to the boundary, in the sense that $d \geq \delta_i/4$ for at least one of them, let y be a point in the boundary such that $d(x_i, y) < 8d$ for both of them (for example, in the case $\delta_1 \leq 4d$, let y be the closest point in the boundary to x_1, so that $d(x_2, y) \leq \delta_1 + d \leq 5d$). Then,

$$\begin{aligned} \left|\frac{u}{v}(x_1) - \frac{u}{v}(x_2)\right| &\leq \left|\frac{u}{v}(x_1) - \frac{u}{v}(y)\right| + \left|\frac{u}{v}(y) - \frac{u}{v}(x_2)\right| \\ &\leq C(|x_1 - y|^\alpha + |y - x_2|^\alpha) \leq 2C(8d)^\alpha \leq 2^{1+3\alpha}C|x_1 - x_2|^\alpha. \end{aligned}$$

Case 3. When the points are close, i.e. $d < 1/16$ and $d < \min(\delta_1, \delta_2)/4$, suppose without loss of generality $0 < \delta_1 \leq \delta_2$. Let

$$r = d(x_2, \partial(B_{3/4} \cap \Omega)) = \min\{3/4 - |x_2|, \delta_2\} \geq \min\{1/4, \delta_2\}.$$

Now, we introduce an auxiliary function $w = u - \mu v$, with μ to be determined later.

$$\left|\frac{u}{v}(x_1) - \frac{u}{v}(x_2)\right| \leq \frac{v(x_1)|w(x_1) - w(x_2)| + |w(x_2)||v(x_1) - v(x_2)|}{v(x_1)v(x_2)}.$$

Hence, since $\mathcal{L}w = f - \mu g$, $\mathcal{L}v = g$, by interior regularity estimates,

$$\begin{aligned}|w(x_1) - w(x_2)| &\leq C_1|x_1 - x_2|^{\alpha'}\left(r^{-\alpha'}\|w\|_{L^\infty(B_{r/2}(x_2))} + r^{2-n/q-\alpha'}\|\mathcal{L}w\|_{L^q(B_{r/2}(x_2))}\right)\\ &\leq C_1|x_1 - x_2|^{\alpha'}\left(r^{-\alpha'}\|w\|_{L^\infty(B_{r/2}(x_2))} + (1+|\mu|)c_0 r^{2-n/q-\alpha'}\right),\\ |v(x_1) - v(x_2)| &\leq C_1|x_1 - x_2|^{\alpha'}\left(r^{-\alpha'}\|v\|_{L^\infty(B_{r/2}(x_2))} + r^{2-n/q-\alpha'}\|g\|_{L^q(B_{r/2}(x_2))}\right)\\ &\leq C_1|x_1 - x_2|^{\alpha'}\left(r^{-\alpha'}\|v\|_{L^\infty(B_{r/2}(x_2))} + c_0 r^{2-n/q-\alpha'}\right).\end{aligned}$$

We may assume without loss of generality that $\alpha' \in (0, 2-\kappa)$.

Now, by the interior Harnack inequality and Corollary 4.3.11, tweaking the constants, $v(x_i) \leq \|v\|_{L^\infty(B_{r/2}(x_2))} \leq Cv(x_i)$. Then, combining our estimates,

$$\begin{aligned}&\frac{1}{C_1}\left|\frac{u}{v}(x_1) - \frac{u}{v}(x_2)\right||x_1 - x_2|^{-\alpha'} \leq\\ &\leq \frac{\|w\|_{L^\infty(B_{r/2}(x_2))} + (1+|\mu|)c_0 r^{2-n/q}}{v(x_2)r^{\alpha'}} + \frac{|w(x_2)|(\|v\|_{L^\infty(B_{r/2}(x_2))} + c_0 r^{2-n/q})}{v(x_1)v(x_2)r^{\alpha'}}\\ &\leq \frac{C\|w\|_{L^\infty(B_{r/2}(x_2))}}{r^{\alpha'}\|v\|_{L^\infty(B_{r/2}(x_2))}} + \frac{C(1+|\mu|)c_0 r^{2-n/q}}{r^{\alpha'}\|v\|_{L^\infty(B_{r/2}(x_2))}} + \frac{C^2\|w\|_{L^\infty(B_{r/2}(x_2))}}{r^{\alpha'}\|v\|_{L^\infty(B_{r/2}(x_2))}} +\\ &\quad + \frac{C^2\|w\|_{L^\infty(B_{r/2}(x_2))}c_0 r^{2-n/q}}{r^{\alpha'}\|v\|^2_{L^\infty(B_{r/2}(x_2))}}.\end{aligned}$$

Now we distinguish two cases: when $r = 1/4$ we just use the global estimates, and when $r < 1/4$ we do some finer computations.

Case 3.1. When $r = 1/4$, let $\mu = 0$. Hence $w = u$. Since $-C'v \leq u \leq C'v$ in $B_{3/4} \cap \Omega$, $\|w\|_{L^\infty(B_{r/2}(x_2))} \leq C'\|v\|_{L^\infty(B_{r/2}(x_2))}$, and $\|v\|_{L^\infty(B_{r/2}(x_2))} \geq c_1 r^\kappa$ by Corollary 4.3.11, then the right hand side of the previous inequality is bounded by some constant C_2 that only depends on n, q, λ, Λ. Hence,

$$\left|\frac{u}{v}(x_1) - \frac{u}{v}(x_2)\right| \leq C_1 C_2 |x_1 - x_2|^{\alpha'}.$$

Case 3.2. If $r < 1/4$, $r = \delta_2$. Choose $\mu = u(x_2)/v(x_2)$, so that $w(x_2) = 0$. Let k_0 be the maximum positive integer such that $\delta_2 < 2^{-k_0}$ (hence $\delta_2 \geq 2^{-k_0-1}$). Then, $k_0 \geq 2$, $d < \delta_2/4$, and x_1, x_2 belong to $\Omega \cap B_{2^{-k_0+1}}(y)$, with $y \in \partial\Omega$, for instance, the closest point in $\partial\Omega$ to x_2 ($d(y, x_1) < \delta_2 + d < 2\delta_2 < 2^{-k_0+1}$ by the triangle inequality). For the same reason, $B_r(x_2) \subset \Omega \cap B_{2^{-k_0+1}}(y)$.

By the estimate (4.11), $\|w\|_{L^\infty(B_r(x_2))} \leq (2C')(1-\theta)^{k_0-2}\|v\|_{L^\infty(B_r(x_2))}$, and combining it with the previous result and the fact that $1 - \theta = 1/2^\alpha$,

$$\begin{aligned}\frac{1}{C_1}\left|\frac{u}{v}(x_1) - \frac{u}{v}(x_2)\right||x_1 - x_2|^{-\alpha'} \leq &\frac{2C'(C + C^2)(1-\theta)^{k_0-2}}{r^{\alpha'}} +\\ &+ \frac{(C(1+|\mu|) + C^2(2C'(1-\theta)^{k_0-2}))c_0 r^{2-n/q-\alpha'}}{\|v\|_{L^\infty(B_{r/2}(x_2))}}.\end{aligned}$$

We put all the constants (everything that does not depend on r, k_0) together, and notice that $|\mu| \leq C'$ and $2^{-k_0-1} \leq r < 2^{-k_0}$. Additionally, we dismiss the term $(1-\theta)^{k_0-2} \leq 1/(1-\theta)^2$ as a constant in the second fraction. Simplifying, we get

$$\left|\frac{u}{v}(x_1) - \frac{u}{v}(x_2)\right| \le |x_1 - x_2|^{\alpha'} \left(2^{k_0(\alpha'-\alpha)}C_3 + \frac{2^{k_0(n/q+\alpha'-2)}C_4}{\|v\|_{L^\infty(B_{r/2}(x_2))}}\right).$$

Since $\|v\|_{L^\infty(B_{r/2}(x_2))} \ge c_1 r^\kappa = c_1\delta_2^\kappa \ge c_1 2^{-\kappa(k_0+1)}$,

$$\frac{2^{k_0(n/q+\alpha'-2)}C_4}{\|v\|_{L^\infty(B_{r/2}(x_2))}} \le 2^{k_0(n/q+\alpha'+\kappa-2)+1}C_4/c_1 \le 2C_4/c_1.$$

If $\alpha \ge \alpha'$, $|x_1 - x_2|^{\alpha'} 2^{k_0(\alpha'-\alpha)} \le |x_1 - x_2|^{\alpha'}$. If $\alpha < \alpha'$, take into account that $r = |x_1 - x_2| < 2^{-k_0}$, and then

$$|x_1 - x_2|^{\alpha'} 2^{k_0(\alpha'-\alpha)} = \left(\frac{|x_1 - x_2|}{2^{-k_0}}\right)^{\alpha'} 2^{-k_0\alpha} \le \left(\frac{|x_1 - x_2|}{2^{-k_0}}\right)^{\alpha} 2^{-k_0\alpha} = |x_1 - x_2|^\alpha.$$

In either case,

$$|x_1 - x_2|^{\alpha'} 2^{k_0(\alpha'-\alpha)} \le |x_1 - x_2|^{\min\{\alpha,\alpha'\}}.$$

Hence,

$$\left|\frac{u}{v}(x_1) - \frac{u}{v}(x_2)\right| \le C_5|x_1 - x_2|^{\min\{\alpha,\alpha'\}}.$$

Observe that we have proved that $|(u/v)(x_1) - (u/v)(x_2)| \le C|x_1 - x_2|^\alpha$ for various values of $C, \alpha > 0$. For the expression to be always valid, take the maximum multiplicative constant and the minimum exponent. □

4.4 The boundary Harnack in slit domains

We also consider our problem in *slit domains*, as introduced in [72, 73]. We define them in the unit ball B_1 to keep the notation uncluttered.

Definition 4.4.1. We say Ω is a slit domain with Lipschitz constant L if $\Omega = B_1 \setminus K$, with K a closed subset of the graph of a Lipschitz function $g : B_1' \to \mathbb{R}$, with $g(0) = 0$:

$$\Omega = B_1 \setminus K, \quad K \subset \Gamma := \{(x', x_n) \in B_1' \times \mathbb{R} : x_n = g(x')\}, \quad \|g\|_{C^{0,1}} = L.$$

Additionally, we define the upper and lower *halves* of Ω,

$$\Omega^+ = \Omega \cap \{(x', x_n) \in B_1 : x_n \ge g(x')\}, \quad \Omega^- = \Omega \cap \{(x', x_n) \in B_1 : x_n \le g(x')\}.$$

We will write $\Omega^\pm$ to refer to Ω^+ or Ω^- indistinctly.

An analogous reasoning to the proof of Theorem 4.1.2 for slit domains yields the following result.

Theorem 4.4.2. *Let $q > n$ and let $\mathcal{L}$ be as in (4.2) or (4.3). There exist small constants $c_0 > 0$ and $L_0 > 0$ such that the following holds.*

Let $\Omega = B_1 \setminus K$ *be a slit domain as in Definition 4.4.1, with Lipschitz constant* $L < L_0$. *Let* u *and* $v > 0$ *be* L^n*-viscosity or weak solutions of*

$$\begin{cases} \mathcal{L}u = f & \text{in } B_1 \setminus K \\ u = 0 & \text{on } K \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}v = g & \text{in } B_1 \setminus K \\ v = 0 & \text{on } K, \end{cases}$$

with f *and* g *satisfying (4.4).*

Additionally, assume that $v(e_n/2) \geq 1$, $v(-e_n/2) \geq 1$, *and either* $u > 0$ *in* $B_1 \setminus K$ *and* $\max\{u(e_n/2), u(-e_n/2)\} \leq 1$, *or* $\|u\|_{L^p(B_1)} \leq 1$ *for some* $p > 0$. *Then,*

$$u \leq Cv \quad \text{in} \quad B_{1/2} \setminus K,$$

and

$$\left\| \frac{u}{v} \right\|_{C^{0,\alpha}(\overline{\Omega^\pm} \cap B_{1/2})} \leq C.$$

The positive constants C, c_0, L_0 *and* α *depend only on the dimension,* q, λ, Λ, *as well as* p *and* σ, *when applicable.*

When both functions are positive, we recover the symmetric version of the boundary Harnack.

Corollary 4.4.3. *Let* $q > n$ *and* $\mathcal{L}$ *as in (4.2) or (4.3). There exist small constants* $c_0 > 0$ *and* $L_0 = L_0(q, n, \lambda, \Lambda) > 0$ *such that the following holds.*

Let $\Omega = B_1 \setminus K$ *be a slit domain as in Definition 4.4.1, with Lipschitz constant* $L < L_0$. *Let* u, v *be positive* L^n*-viscosity or weak solutions of*

$$\begin{cases} \mathcal{L}u = f & \text{in } B_1 \setminus K \\ u = 0 & \text{on } K \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}v = g & \text{in } B_1 \setminus K \\ v = 0 & \text{on } K, \end{cases}$$

with f *and* g *satisfying (4.4).*

Assume $\min\{v(e_n/2), v(-e_n/2)\} \geq 1$ *and* $\min\{u(e_n/2), u(-e_n/2)\} \geq 1$. *Then,*

$$C^{-1} \frac{\min\{u(e_n/2), u(-e_n/2)\}}{\max\{v(e_n/2), v(-e_n/2)\}} \leq \frac{u}{v} \leq C \frac{\max\{u(e_n/2), u(-e_n/2)\}}{\min\{v(e_n/2), v(-e_n/2)\}} \quad \text{in} \quad B_{1/2} \setminus K,$$

and

$$\left\| \frac{u}{v} \right\|_{C^{0,\alpha}(\overline{\Omega^\pm} \cap B_{1/2})} \leq C.$$

The positive constants C, c_0, L_0 *and* α *depend only on the dimension,* q, λ, Λ, *as well as* σ, *when applicable.*

Most of the proofs are identical to the one-sided theorem, because we can prove the results for each *side* of Γ and then put them together. There are two exceptions: Proposition 4.3.8 and Lemma 4.3.9. The proof of the proposition is even easier, taking $v = u^+$ and extending it by 0 on K, we are ready to apply Theorem 4.2.6 and see that v is universally bounded.

As for the lemma, we write here an adapted version and the step of the proof that needs to be changed.

Lemma 4.4.4. *Let $q > n$, $\kappa > 1$, and let $\mathcal{L}$ be as in (4.2) or (4.3). There exists $L_* = L_*(q, n, \kappa, \lambda, \Lambda)$ such that the following holds.*

Let $\Omega = B_1 \setminus K$, with $K \subset \Gamma$, be a slit domain as in Definition 4.4.1 with constant $L < L_*$. Let f such that $\|f\|_{L^q(B_1)} \leq c$. Let u be a L^n-viscosity or weak solution of

$$\begin{cases} \mathcal{L}u = f & \text{in } B_1 \setminus K \\ u = 0 & \text{on } K, \end{cases} \quad \text{with} \quad \begin{cases} u \geq 1 & \text{in } \Omega \cap \{x \in B_1 : d(x, \Gamma) > \delta\} \\ u \geq -\varepsilon & \text{in } \Omega \cap B_1. \end{cases} \tag{4.12}$$

Then,

$$\begin{cases} u \geq \rho^\kappa & \text{in } \Omega \cap \{x \in B_\rho : d(x, \Gamma) > \rho\delta\} \\ u \geq -\rho^\kappa \varepsilon & \text{in } B_\rho \end{cases}$$

for some sufficiently small $\rho, \varepsilon, \delta, c \in (0, 1)$, with $\rho > 2\delta$, only depending on the dimension, κ, q, λ, Λ, as well as σ, when applicable.

Proof. The proof of the first inequality is completely analogous to the proof of Lemma 4.3.9.

For the second inequality, we do the same reasoning as in the one-sided case, but now, instead of picking a downwards cone C_z with vertex at $\partial\Omega$, for each x_0 such that $d(x_0, \Gamma) \leq \delta$, we take $z = x_0 - 5\delta/2e_n$. Since Γ is a Lipschitz graph with Lipschitz constant $L < 1/16$, $d(z, \Gamma) \geq 5\delta/(2\sqrt{L^2 + 1}) - \delta > \delta$, so again z and the analogous downwards cone C_z lie in the region where $u \geq \rho^\kappa$. Moreover, $|C_z \cap B_{3\delta}(x_0)| = c_L|B_{3\delta}|$. The rest of the proof continues analogously. □

4.5 Applications to free boundary problems

4.5.1 $\mathcal{C}^{1,\alpha}$ regularity of the free boundary in the obstacle problem

Consider the classical obstacle problem (4.5) in B_1, with $f \geq \tau_0 > 0$, $f \in W^{1,q}$, and assume that 0 is a free boundary point. We will show that we can extend the proof of the $\mathcal{C}^{1,\alpha}$ regularity of the free boundary due to Caffarelli [36] to the case $f \in W^{1,q}$ thanks to our new result. We generalize the steps of the proof in [97, Section 5.4].

Our starting point will be the existence of a *regular* blow-up. We will also take for granted the following nondegeneracy condition: if $x_0 \in \overline{\{u > 0\}}$,

$$\sup_{B_r(x_0)} u \geq cr^2,$$

which follows easily from the fact $f \geq \tau_0 > 0$; see [97, Proposition 5.9].

Proposition 4.5.1. *Let u be a solution of (4.5), with $f \in W^{1,n}$ and $f \geq \tau_0 > 0$. Assume that 0 is a regular free boundary point as in Definition 4.1.5.*

Then, for every $L_0 > 0$ there exists $r > 0$ such that the free boundary is the graph of a Lipschitz function in B_r with Lipschitz constant $L < L_0$.

We will denote

$$u_r(x) := \frac{u(rx)}{r^2}.$$

Observe that the blow-up hypothesis implies that for all $\varepsilon > 0$, there exists r_0 such that

$$\left| u_{r_0} - \frac{\gamma}{2}(x \cdot e)_+^2 \right| < \varepsilon \quad \text{in} \quad B_1,$$

and

$$\left| \partial_\nu u_{r_0} - \gamma (x \cdot e)_+ (x \cdot \nu) \right| < \varepsilon \quad \text{in} \quad B_1$$

for all $\nu \in \mathbb{S}^{n-1}$.

To prove that the free boundary is Lipschitz, we will use the interior and exterior cone conditions, and to do this we will prove that $\partial_\nu u_r \geq 0$, with ν a unit vector, when $\nu \cdot e > c(L)$, where $c(L)$ is the positive constant that ensures that the cone $\{x \in \mathbb{R}^n : x \cdot e = |x| c(L)\}$ has Lipschitz constant L. We need a positivity lemma.

Lemma 4.5.2. *Let u be a solution of (4.5) with $f \in W^{1,n}(B_1)$, $r > 0$ and $\Omega = \{u_r > 0\}$. Let $w = \partial_\nu u_r$. Then, w is a solution of*

$$\begin{cases} \Delta w &= g \quad \text{in } \Omega \cap B_1 \\ w &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

with $g(x) = r \partial_\nu f(rx)$. Assume that, denoting $N_\delta = \{x \in B_1 : d(x, \partial\Omega) < \delta\}$, we have

$$w > -\varepsilon \quad \text{in} \quad N_\delta \quad \text{and} \quad w > M \quad \text{in} \quad \Omega \setminus N_\delta, \tag{4.13}$$

with positive ε and M. Then, $w \geq 0$ in $\Omega \cap B_{1/2}$, provided that ε, r and $\delta > 0$ are small enough.

Proof. First, it is clear that $w > 0$ in $\Omega \setminus N_\delta$. Suppose there exists $x_0 \in B_{1/2} \cap N_\delta$ such that $w(x_0) < 0$. We will arrive at a contradiction using the maximum princple, combined with the ABP estimate, with the function

$$v(x) = w(x) - \eta \left(u_r(x) - \frac{f(x_0)}{2n} |x - x_0|^2 \right).$$

Consider the set $\Omega \cap B_{1/4}(x_0)$. On $\partial\Omega$, $u_r = 0$, hence $v \geq 0$. On $\partial B_{1/4}(x_0) \cap N_\delta$,

$$v(x) \geq -\varepsilon - \eta\delta \|u_r\|_{C^1} + \frac{\eta}{32n}.$$

On $\partial B_{1/4}(x_0) \cap (\Omega \setminus N_\delta)$,

$$v(x) \geq M - \eta \|u_r\|_{C^1}.$$

Notice that $\|u_r\|_{C^1}$ is uniformly bounded as $r \to 0$. Hence, choosing η small enough, the second inequality implies $v \geq M/2$. For the first inequality, choosing now small enough ε and δ, we obtain $v \geq \eta/(64n)$.

This function satisfies $\Delta v(x) = g(x) - \eta(f(x) - f(x_0))$, hence, by the ABP estimate,

$$v(x_0) \geq \min\{M/2, \eta/(64n)\} - C \|g(x) - \eta(f(rx) - f(rx_0))\|_{L^n(B_{1/4}(x_0))}.$$

We estimate g and $f - f(x_0)$ separately. Using the scaling of the L^n norm and taking $r \to 0$,

$$\|g\|_{L^n(B_{1/4}(x_0))} = \|\partial_\nu f\|_{L^n(B_{r/4}(rx_0))} \to 0.$$

On the other hand, by the Poincaré inequality,

$$\|f(rx) - f(rx_0)\|_{L^n(B_{1/4}(x_0))} = \frac{\|f - f(x_0)\|_{L^n(B_{r/4}(rx_0))}}{r} \leq C \|\nabla f\|_{L^n(B_{r/4}(rx_0))} \to 0.$$

Hence, choosing r small enough, ve can have $v(x_0) \geq \min\{M/2, \eta/(64n)\}/2$, which contradicts $v(x_0) < 0$. □

Using the lemma, we prove that there is an arbitrarily wide cone of directions where $\partial_\nu u_r \geq 0$, for small $r > 0$.

Proof of Proposition 4.5.1. We only need to check that, for any $\nu \in \mathbb{S}^{n-1}$ such that $\nu \cdot e > c(L)$, the hypotheses of the lemma hold. By construction, we only need to check that (4.13) holds for a small enough $r > 0$.

Let $\delta = \varepsilon^{1/8}$. By the blow-up, there exists $r > 0$ such that

$$\left| u_r - \frac{\gamma}{2}(x \cdot e)_+^2 \right| < \varepsilon,$$

and

$$\left| \partial_\nu u_r - \gamma(x \cdot e)_+(x \cdot \nu) \right| < \varepsilon.$$

Hence, if $\varepsilon > 0$ is small enough,

$$u_r > \frac{\gamma}{2}(x \cdot e)_+^2 - \varepsilon > \frac{\gamma}{2}\delta^4 - \varepsilon = \frac{\gamma}{2}\varepsilon^{1/2} - \varepsilon > 0 \quad \text{in} \quad \{x \cdot e > \delta^2\}.$$

Moreover,

$$u_r = 0 \quad \text{in} \quad \{x \cdot e < -\delta^2\},$$

as we prove by contradiction from the nondegeneracy. Suppose $u_r(y) > 0$ for some y such that $y \cdot e < -\delta$. Then,

$$\sup_{B_{\delta^2}(y)} u_r \geq c\delta^4 = c\varepsilon^{1/2},$$

but since $B_{\delta^2}(y) \subset \{x \cdot e < 0\}$, $u_r < \varepsilon$, which cannot happen if ε is small enough. Hence, the free boundary is contained in the strip $\{|x \cdot e| < \delta^2\}$.

Now, let a unit ν such that $\nu \cdot e > c(L)$. The lower bound for $\partial_\nu u_r$ in N_δ only takes into account the convergence of the blow-up,

$$\partial_\nu u_r > \gamma c(L)(x \cdot e)_+ - \varepsilon \geq -\varepsilon.$$

On the other hand, if $z \in \Omega \setminus N_\delta$, since the free boundary is at a distance lower than δ^2 from the hyperplane $\{x \cdot e = 0\}$, $z \cdot e > \delta - \delta^2$. Hence,

$$\partial_\nu u_r(z) > \gamma c(L)(z \cdot e)_+ - \varepsilon > \gamma c(L)(\delta - \delta^2) - \varepsilon = \gamma c(L)(\varepsilon^{1/8} - \varepsilon^{1/4}) - \varepsilon =: M,$$

where $M > 0$ provided that ε is small enough.

Notice that r and ε (thus δ) are uniform in ν. Now, applying the previous Lemma 4.5.2, for all unit ν such that $\nu \cdot e > c(L)$, $\partial_\nu u_r \geq 0$, which is equivalent to $\partial_\nu u \geq 0$ in B_r. Now, if $x_0 \in B_r$ is a free boundary point, $u(x_0) = 0$, hence $u(x_0 - t\nu) \leq 0$ whenever $x_0 - t\nu \in B_r$. Since $u \geq 0$, there is a cone *behind* x_0 where $u = 0$, i.e.

$$u = 0 \quad \text{in} \quad B_r \cap \{(x - x_0) \cdot e < -c(L)|x|\}.$$

In the interior of the cone, there are no free boundary points because u is 0 in a neighbourhood of all points. This is the interior cone. To check the exterior cone condition, suppose there is another free boundary point x_1 in the set $B_r \cap \{x_0 + t\nu : \nu \cdot e > c(L), t \in \mathbb{R}^+\}$. Then, by applying the interior cone condition to x_1, we get that x_0 cannot be a free boundary point, a contradiction. This proves that, in B_r, the free boundary is a Lipschitz graph with constant L in the direction e. □

Now we can use our new boundary Harnack inequality to prove the $\mathcal{C}^{1,\alpha}$ regularity of the free boundary at regular points *à la* Caffarelli. To do this, we must ask the right hand side to belong to $W^{1,q}$ with $q > n$, which is slightly more restrictive and implies that f is Hölder continuous.

Proof of Corollary 4.1.6. As it is customary in this kind of proof, we will use the boundary Harnack with the derivatives of u_r. Let $L = L_0(q, n, 1, 1)/2$ with the L_0 defined in Corollary 4.1.4. From Proposition 4.5.1, there exists $r > 0$ such that the free boundary is a Lipschitz graph with constant L in B_r. Assume without loss of generality that the direction of the graph is $e = e_n$, and that $L < 1$.

For $i = 1, \dots, n-1$, consider the functions

$$w_1 = \partial_i u_r \quad \text{and} \quad w_2 = \partial_n u_r.$$

They are both solutions of $\Delta w_j = g_j$, with $g_1(x) = r\partial_i f(rx)$, $g_2(x) = r\partial_n f(rx)$. Moreover, w_2 is positive. To be able to use the boundary Harnack, we need to see that the right hand is small. Indeed, taking $r \to 0$,

$$\|g_j\|_{L^q(B_1)} \leq \|r\nabla f(rx)\|_{L^q(B_1)} = 2r^{1-n/q}\|\nabla f\|_{L^q(B_r)} \to 0.$$

Finally, by the blow-up convergence,

$$w_j(e_n/2) > \gamma/2 - \varepsilon > \gamma/4, \quad w_j(e_n/2) < \gamma/2 + \varepsilon < \gamma.$$

Thus, we can normalize w_j dividing by $w_j(e_n/2)$ and the right hand side still converges to 0 in norm.

Let $\Omega_r = \{u_r > 0\}$. By the boundary Harnack with right hand side, Theorem 4.1.2,

$$\frac{w_1}{w_2} \in \mathcal{C}^{0,\alpha}(B_{1/2} \cap \Omega_r) \quad \Rightarrow \quad \frac{\partial_i u_r}{\partial_n u_r} \in \mathcal{C}^{0,\alpha}(B_{1/2} \cap \Omega_r).$$

The unit normal vector to any level set $\{u_r = t\}$, $t > 0$, is, by components,

$$\hat{n}^i = \frac{\partial_i u_r}{|\nabla u_r|} = \frac{\partial_i u_r/\partial_n u_r}{\sqrt{1 + \sum_{j=1}^{n-1}(\partial_j u_r/\partial_n u_r)^2}} \in \mathcal{C}^{0,\alpha}(B_{1/2} \cap \Omega_r).$$

As this expression is $\mathcal{C}^{0,\alpha}$ up to the boundary, this proves the normal vector to the free boundary is $\mathcal{C}^{0,\alpha}$, and by a simple calculation it follows that the free boundary is $\mathcal{C}^{1,\alpha}$. □

4.5.2 $\mathcal{C}^{1,\alpha}$ regularity of the free boundary in the fully nonlinear obstacle problem

Consider the fully nonlinear obstacle problem in the general version (4.7), under the assumptions in Corollary 4.1.7.

Our starting point will be the existence of a regular blow-up in the sense of Definition 4.1.5, i.e., there exists $r_k \downarrow 0$ such that

$$\frac{u(r_k x)}{r_k^2} \to \frac{\gamma}{2}(x \cdot e)_+^2 \quad \text{in } \mathcal{C}^1_{\text{loc}}(\mathbb{R}^n)$$

for some $\gamma > 0$ and $e \in \mathbb{S}^{n-1}$. We also need the classical nondegeneracy condition: if $x_0 \in \overline{\{u > 0\}}$,

$$\sup_{B_r(x_0)} u \geq cr^2.$$

From here, we will extend the proof of [120] to the case where $f \in W^{1,q}$ (and not necessarily Lipschitz), and we will also prove $\mathcal{C}^{1,\alpha}$ regularity of the free boundary.

Our first step is an analogue to [120, Lemma 3.7] for the case $f \in W^{1,n}$.

Lemma 4.5.3. *Let u be a solution of*

$$\begin{cases} F(D^2u(x), rx) &= f(rx)\chi_{\{u>0\}} \quad a.e.\ in\ B_1 \\ u &\geq 0 \quad a.e.\ in\ B_1. \end{cases}$$

Assume that the conditions (H1), (H2) and (H3) from Corollary 4.1.7 hold. If

$$C_0\partial_\nu u - u \geq -\varepsilon \quad in \quad B_1,$$

for any $C_0 > 0$, then

$$C_0\partial_\nu u - u \geq 0 \quad in \quad B_{1/2},$$

provided that $r, \varepsilon > 0$ are sufficiently small.

The proof is the same as in our source, except for the final step. We provide it here for the convenience of the reader.

Remark 4.5.4. For this lemma, the case $q = n$, i.e., when $f \in W^{1,n}$ and $F \in W^{1,n}$ with respect to the second variable, is also true.

Proof. Let $x \in \{u > 0\}$ and $\partial_1 F(M, x)$ denote the subdifferential of F at the point (M, x) with respect to the first variable. Since F is convex in M, then $\partial_1 F(M, x) \neq \varnothing$. Consider a measurable function P^M with $P^M(x) \in \partial_1 F(M, x)$. Since $f \in \mathcal{C}^\alpha$, by interior regularity estimates $u \in \mathcal{C}^{2,\alpha}_{\text{loc}}(\{u > 0\})$, and thus we can define the measurable coefficients

$$a_{ij}(x) := (P^{D^2u(x)}(rx))_{ij} \in \partial_1 F(D^2u(x), rx).$$

Since F is convex in the first variable and $F(0, x) \equiv 0$, then for any unit vector ν,

$$\sum_{i,j=1}^n a_{ij}(x)\frac{\partial_{ij}u(x + h\nu) - \partial_{ij}u(x)}{h} \leq \frac{F(D^2u(x + h\nu), rx) - F(D^2u(x), rx)}{h},$$

$$\sum_{i,j=1}^n a_{ij}(x)\partial_{ij}u(x) = F(0, rx) + a_{ij}\partial_{ij}u(x) \geq F(D^2u(x), rx) = f(rx),$$

provided that $x + h\nu \in \{u > 0\} \cap B_1$. Now, since uniform limits of L^n-viscosity solutions are L^n-viscosity solutions ([38, Theorem 3.8]),

$$\begin{aligned} \sum_{i,j=1}^n a_{ij}(x)\partial_{ij}\partial_\nu u(x) &\leq \limsup_{h\to 0} \frac{F(D^2u(x + h\nu), rx) - F(D^2u(x), rx)}{h} \\ &= \limsup_{h\to 0} \frac{F(D^2u(x + h\nu), rx) - f(rx)}{h} \\ &= \limsup_{h\to 0} \frac{F(D^2u(x + h\nu), rx) - F(D^2u(x + h\nu), rx + rh\nu)}{h} + \\ &\quad + \frac{f(rx + rh\nu) - f(rx)}{h} \\ &= r(\partial_\nu f)(rx) - r(\partial_{2,\nu}F)(D^2u(x), rx) \end{aligned}$$

in the L^n-viscosity sense.

Suppose there exists $y_0 \in \Omega \cap B_{1/2}$ such that $C_0\partial_\nu u(y_0) - u(y_0) < 0$. Then, we consider the auxiliary function

$$w(x) = C_0\partial_\nu u(x) - u(x) + \tau_0\frac{|x-y_0|^2}{4n\Lambda}.$$

Then,

$$\begin{aligned} a_{ij}\partial_{ij}w(x) &\leq rC_0(\partial_\nu f)(rx) - rC_0(\partial_{2,\nu}F)(D^2u(x), rx) - f(rx) + \tau_0/2 \\ &\leq rC_0(\partial_\nu f(rx) - \partial_{2,\nu}F(D^2u(x), rx)) - f(rx) + \tau_0/2 \\ &\leq rC_0(\partial_\nu f(rx) - \partial_{2,\nu}F(D^2u(x), rx)) =: rR(rx). \end{aligned}$$

Hence, by the ABP estimate, since $R \in L^n(B_1)$,

$$0 > \inf_{\Omega\cap B_{1/4}(y_0)} w \geq \inf_{\partial(\Omega\cap B_{1/4}(y_0))} w - C\|rR(rx)\|_{L^n(\Omega\cap B_{1/4}(y_0))}.$$

By the scaling of the L^n norm, the second term in the sum is bounded by

$$C\|R\|_{L^n(B_{r/4}(ry_0))} \to 0$$

as $r \to 0$. On the other hand, $w \equiv 0$ on $\partial\Omega$, and

$$w \geq -\varepsilon + \frac{\tau_0}{64n\Lambda} \quad \text{on} \quad \Omega \cap \partial B_{1/4}.$$

Therefore, choosing ε and r small enough we reach $w > 0$ in $\Omega \cap B_{1/4}(y_0)$, a contradiction. □

Now, as we show next, by the $\mathcal{C}^1$ convergence of the blow-up we can fulfill the sufficient conditions in Lemma 4.5.3, and prove that the free boundary is Lipschitz at regular points, analogously to Proposition 4.5.1. Then, applying the boundary Harnack inequality, we can improve the regularity up to $\mathcal{C}^{1,\alpha}$.

We denote $u_r(x) := r^{-2}u(rx)$ as in Proposition 4.5.1.

Proof of Corollary 4.1.7. Let

$$u_0(x) = \frac{\gamma}{2}(x\cdot e)_+^2$$

be the blow-up at 0. Let $s \in (0,1)$. Then,

$$\frac{\partial_\nu u_0}{s} - u_0 = \gamma\left(\frac{(x\cdot e)_+(e\cdot\nu)}{s} - \frac{(x\cdot e)_+^2}{2}\right) \geq 0$$

for any direction $\nu \in \mathbb{S}^{n-1}$ such that $\nu\cdot e \geq s/2$. From the $\mathcal{C}^1$ convergence of the blow-up, there exists r_k such that

$$\frac{\partial_\nu u_{r_k}}{s} - u_{r_k} \geq -\varepsilon \quad \text{in} \quad B_1.$$

By Lemma 4.5.3, this implies

$$\frac{\partial_\nu u_{2\rho}}{s} - u_{2\rho} \geq 0 \quad \text{in} \quad B_{1/2},$$

for some sufficiently small $\rho > 0$.

In particular, this shows that the free boundary fulfills the interior and exterior cone conditions in B_ρ and therefore it is Lipschitz, with Lipschitz constant $L(s)$, that satisfies $L(s) \to 0$ as $s \to 0$.

Now, assume without loss of generality $e = e_n$. For $i = 1, \ldots, n-1$, consider the functions

$$w_1 = \partial_i u_\rho \quad \text{and} \quad w_2 = \partial_n u_\rho.$$

Notice that $w_2 \geq 0$. Since F is $\mathcal{C}^1$ with respect to D^2u and $F(D^2u, x) \in W^{1,q}$, then $u \in W^{3,q}$ and we can commute the third derivatives as follows,

$$\begin{aligned}\partial_\nu F(D^2 u_\rho(x), \rho x) = \sum_{i,j=1}^{n} F_{ij}\partial_\nu \partial_{ij} u_\rho + \rho \partial_{2,\nu} F = \sum_{i,j=1}^{n} F_{ij}\partial_{ij}(\partial_\nu u_\rho) + \rho\partial_{2,\nu}F = \rho\partial_\nu f,\\ \mathcal{L}(\partial_\nu u_\rho) = \rho(\partial_\nu f - \partial_{2,\nu}F).\end{aligned}$$

Here, $\mathcal{L}w = \mathrm{Tr}(A(x)w)$, with $A(x) = (F_{ij}(D^2u_\rho, \rho x))_{ij}$. Hence, w_1 and w_2 are both solutions of

$$\mathcal{L}w_1 = g_1 := \rho(\partial_i f - \partial_{2,i}F)(\rho x) \quad \text{and} \quad \mathcal{L}w_2 = g_2 := \rho(\partial_n f - \partial_{2,n}F)(\rho x).$$

To be able to use the boundary Harnack, we need to show that the right hand is small. Indeed, taking $\rho \to 0$,

$$\|g_j\|_{L^q(B_1)} \leq \|\rho(\nabla f(\rho x) + \nabla_2 F(\rho x))\|_{L^q(B_1)} = \rho^{1-n/q}\|\nabla f + \nabla_2 F\|_{L^q(B_\rho)} \to 0.$$

Finally, by the blow-up convergence,

$$w_j(e_n/2) > \gamma/2 - \varepsilon > \gamma/4, \quad w_j(e_n/2) < \gamma/2 + \varepsilon < \gamma.$$

Thus, we can normalize w_j dividing by $w_j(e_n/2)$ and the right hand side still converges to 0 in norm.

Let $\Omega_\rho = \{u_\rho > 0\}$. By the boundary Harnack with right hand side, Theorem 4.1.2,

$$\frac{w_1}{w_2} \in \mathcal{C}^{0,\alpha}(\Omega_\rho \cap B_{1/2}) \quad \Rightarrow \quad \frac{\partial_i u_\rho}{\partial_n u_\rho} \in \mathcal{C}^{0,\alpha}(\Omega_\rho \cap B_{1/2}).$$

The unit normal vector to any level set $\{u_\rho = t\}$, $t > 0$, is, by components,

$$\hat{n}^i = \frac{\partial_i u_\rho}{|\nabla u_\rho|} = \frac{\partial_i u_\rho / \partial_n u_\rho}{\sqrt{1 + \sum_{j=1}^{n-1}(\partial_j u_\rho/\partial_n u_\rho)^2}} \in \mathcal{C}^{0,\alpha}(\Omega_\rho \cap B_{1/2}).$$

As this expression is $\mathcal{C}^{0,\alpha}$ up to the boundary, this proves the normal vector to the free boundary is $\mathcal{C}^{0,\alpha}$, and it follows that the free boundary is $\mathcal{C}^{1,\alpha}$. □

4.5.3 $\mathcal{C}^{1,\alpha}$ regularity of the free boundary in the fully nonlinear thin obstacle problem

Recall the fully nonlinear thin obstacle problem (4.9), under the assumptions in Corollary 4.1.9.

We will denote $u = v - \varphi$. In this case, we know the following.

Proposition 4.5.5 ([168]). *Assume that 0 is a regular free boundary point for (4.9), where F is uniformly elliptic, convex and $F(0) = 0$, and $\varphi \in \mathcal{C}^{1,1}$. Then, there exists $e \in \mathbb{S}^{n-1} \cap \{x_n = 0\}$ such that for any $L > 0$ there exists $r > 0$ for which*

$$\partial_\nu u \geq 0 \quad in \quad B_r \quad for\ all \quad \nu \cdot e \geq \frac{L}{\sqrt{L^2+1}}, \quad \nu \in \mathbb{S}^{n-1} \cap \{x_n = 0\}.$$

In particular, the free boundary is Lipschitz in B_r, with Lipschitz constant L.

Now, using our new boundary Harnack in slit domains, Theorem 4.4.2, on top of this proposition, we derive the $\mathcal{C}^{1,\alpha}$ regularity of the free boundary at regular points.

Proof of Corollary 4.1.9. Let $\Omega = B_1 \setminus \{(x', 0) : u(x', 0) = 0\}$. The free boundary is a Lipschitz graph inside $B_r \cap \{x_n = 0\}$. Suppose without loss of generality that the direction of the graph is $e = e_{n-1}$. Choosing L and r small enough, for all $\nu \in \mathbb{S}^n \cap \{x_n = 0\}$ such that $\nu \cdot e_{n-1} \geq 1/2$, $\partial_\nu u \geq 0$ in B_r.

For $i = 1, \dots, n-2$, consider the functions

$$w_1 = \partial_i u, \quad w_2 = \partial_{n-1} u.$$

Since $F \in \mathcal{C}^1$ and $F(D^2 v) = 0$, then $v \in W^{3,p}$ for all $p < \infty$ and we can commute the third derivatives as follows,

$$\begin{aligned} &\partial_\nu(F(D^2 v)) = 0 \quad \text{in} \quad \Omega, \\ &\partial_\nu(F(D^2 v)) = \sum_{i,j=1}^n F_{ij}\partial_\nu(\partial_{ij} v) = \sum_{i,j=1}^n F_{ij}\partial_{ij}(\partial_\nu v) = \mathrm{Tr}(AD^2(\partial_\nu v)), \end{aligned}$$

Moreover, w_2 is positive. Then, using that $v = u + \varphi$,

$$\left\{ \begin{array}{rcll} \mathcal{L}w_1 & = & -\mathcal{L}(\partial_i \varphi) & \text{in } \Omega \\ w_1 & = & 0 & \text{on } B_1 \setminus \Omega, \end{array} \right. \quad and \quad \left\{ \begin{array}{rcll} \mathcal{L}w_2 & = & -\mathcal{L}(\partial_{n-1} \varphi) & \text{in } \Omega \\ w_2 & = & 0 & \text{on } B_1 \setminus \Omega. \end{array} \right.$$

where $\mathcal{L}w = \mathrm{Tr}(AD^2 w)$, $A = (F_{ij} \circ D^2 u)_{ij}$. Now, we will check that, after a scaling, $w_2(e_n/2) \geq 1$ and the right hand side becomes arbitrarily small. Define

$$\tilde{w}_2(x) = \frac{w_2(sx)}{s} \quad \text{and} \quad \tilde{\varphi}(x) = \frac{\varphi(sx)}{s^2}.$$

Now, we check that the right hand side is as small as required. Indeed, letting $s_k \to 0$,

$$\begin{aligned} \|\mathcal{L}(\partial_{n-1}\tilde{\varphi})\|_{L^q(B_1)} \leq \Lambda \|D^3 \tilde{\varphi}\|_{L^q(B_1)} &= \Lambda \|s_k \varphi(s_k x)\|_{W^{3,q}(B_1)} \\ &= \Lambda s_k^{1-n/q} \|\varphi\|_{W^{3,q}(B_{s_k})} \to 0. \end{aligned}$$

The right hand side becomes arbitrarily small in the equation for w_1 analogously. Then, since 0 is a regular free boundary point, by the convergence of the blow-up,

$$\tilde{w}_2(e_{n-1}/2) \to \infty$$

for a sequence of values $\{s_k\} \to 0$. Now, by the interior Harnack inequality combined with the ABP estimate, since $\tilde{w}_2 \geq 0$ in Ω and the distance between the segment joining $e_{n-1}/2$ and

$\pm e_n/2$ and the contact set is positive and larger than some constant $c(L,n)$ only depending on the Lipschitz constant of the free boundary and the dimension,

$$\tilde{w}_2(\pm e_n/2) \geq c_1\tilde{w}_2(e_{n-1}/2) - c_2\|\mathcal{L}(\partial_{n-1}\tilde{\varphi})\|_{L^n(B_1)} \geq c_1\tilde{w}_2(e_{n-1}/2) - c_2\Lambda\|\varphi\|_{W^{3,n}},$$

for some positive c_1 and c_2 only depending on the dimension, L, λ and Λ.

Therefore, letting $s_k \to 0$, $\tilde{w}_2(\pm e_n/2) \geq 1$. If $\|\tilde{w}_1\|_{L^1} > 1$, we normalize it (notice that this step can only make the right hand side smaller). Thus, by the boundary Harnack inequality with right hand side for slit domains, Theorem 4.4.2, $w_1/w_2 \in \mathcal{C}^{0,\alpha}$ in $\Omega \cap B_{s/2}$. Thus, $\partial_i u/\partial_{n-1}u \in \mathcal{C}^{0,\alpha}$.

Now, the unit normal vector to any level set in the *thin space* $\{x_n = 0\} \cap \{u = t\}$ with $t > 0$ is, by components,

$$\hat{n}^i = \frac{\partial_i u}{\sqrt{\sum_{j=1}^{n-1}|\partial_j u|^2}} = \frac{\partial_i u/\partial_{n-1}u}{\sqrt{1+\sum_{j=1}^{n-2}(\partial_j u/\partial_{n-1}u)^2}} \in \mathcal{C}^{0,\alpha}$$

Then, letting $t \to 0^+$, we recover that the normal vector to the free boundary is $\mathcal{C}^{0,\alpha}$, and hence the free boundary is a $\mathcal{C}^{1,\alpha}$ graph. □

4.6 Sharpness of the results

We construct two examples that show that:

- Without the smallness assumption on the Lipschitz constant of the domain, Theorem 4.1.2 fails.
- For $q = n$, Theorem 4.1.2 fails.
- For divergence form operators, if the coefficients are only bounded and measurable, Theorem 4.1.2 fails.

As a first observation, see [3], take for instance $\Omega = \{x_n > 0\} \subset \mathbb{R}^n$, and let $u(x) = x_n$, $v(x) = x_n^2$. These functions are normalized in the sense that $u(e_n) = v(e_n) = 1$, and vanish contiuously on $\partial\Omega$. Even in a flat domain, a function with a too large Laplacian, $|\Delta v| = 2$, will never be comparable to a harmonic function near the boundary. Hence, the right hand side of the equation must be small, otherwise the result fails.

The following example in two dimensions shows that if we ask Δv to be small in L^q norm, for any $q > n$ there is a cone narrow enough such that we can find harmonic functions that are not comparable with v. Moreover, if we consider a fixed cone, there exists $q > n$ such that the L^q boundedness of the right hand side is not enough to have a boundary Harnack. If $q = n$, such counterexamples are valid for any cone.

Proposition 4.6.1. *Let $L > 0$, $q > 0$, and assume*

$$\frac{\pi}{2\arctan(1/L)} + \frac{2}{q} > 2. \tag{4.14}$$

Then, for every $\delta > 0$, there exists a cone $\Omega \subset \mathbb{R}^2$ with Lipschitz constant L, and positive functions u, v that vanish continuously on $\partial\Omega$ such that

$$u(0,1) = v(0,1) = 1, \quad \Delta u = 0 \quad \textit{and} \quad \|\Delta v\|_{L^q(\Omega)} < \delta,$$

but

$$\sup_{\Omega} \frac{u}{v} = \infty.$$

In particular, Theorem 4.1.2 fails for $q = n$.

Proof. Consider the cone $\Omega = \{(x, y) \in \mathbb{R}^2 : y > L|x|\}$ and let

$$\beta = \frac{\pi}{2 \arctan(1/L)} \quad \text{and} \quad u(x, y) = \mathrm{Re}\left((-ix + y)^\beta\right).$$

Then, u is harmonic, positive in Ω, and vanishes continuously on $\partial\Omega$. Let ψ be a positive smooth function such that

$$0 \le \psi \le u, \quad \mathrm{Supp}\,\psi \subset B_{1/3}(0, 1) \quad \text{and} \quad \psi(0, 1) = u(0, 1) = 1.$$

Define the scalings $\psi_\varepsilon(x, y) = \varepsilon^\beta \psi(x/\varepsilon, y/\varepsilon)$. Since u is homogeneous of degree β, $0 \le \psi_\varepsilon \le u$. Moreover,

$$\Delta\psi_\varepsilon(x, y) = \varepsilon^{\beta-2}(\Delta\psi)(x/\varepsilon, y/\varepsilon).$$

Now, we construct v as the following infinite sum, that converges uniformly.

$$v := u - \sum_{k=k_0}^{\infty} (1 - 2^{-k})\psi_{2^{-k}}.$$

Since the supports of $\psi_{2^{-k}}$ are disjoint, $v \ge 0$. On the other hand,

$$\|\Delta v\|_{L^q(\Omega)} \le \sum_{k=k_0}^{\infty} \|\Delta\psi_{2^{-k}}\|_{L^q(\Omega)} = \sum_{k=k_0}^{\infty} 2^{-k(\beta-2+2/q)}\|\Delta\psi\|_{L^q(\Omega)} \to 0$$

as $k_0 \to \infty$, since $\beta + 2/q > 2$ by hypothesis. Hence, we can choose k_0 big enough so that $\|\Delta v\|_{L^q(\Omega)} < \delta$.

To end, for $k \ge k_0$,

$$\frac{u(2^{-k}, 0)}{v(2^{-k}, 0)} = 2^k \to \infty,$$

as wanted. □

Remark 4.6.2. Since $\arctan(1/L) \in (0, \pi/2)$, the first term in the condition (4.14) is always greater than 1, hence, if $q \le 2$ there are always counterexamples to the boundary Harnack with right hand side bounded in L^q.

On the other hand, if $L > 1$, $\arctan(1/L) < \pi/4$, and the condition is fulfilled for all $q > 0$ and $q = \infty$.

The limiting case $L = 0$, $q = 2$ (or $q = n$ in higher dimensions) corresponds to domains that are locally a half-space. We have not considered this particular case in our setting.

The existence of such example shows that, to have a boundary Harnack inequality for equations with a right hand side, we need the Lipschitz constant of the boundary to be sufficiently small, and also the right hand side to be small compared to the values of the function. It also shows a trade-off between the maximum possible slope of the boundary and the exponent of the L^q boundedness of the right hand side.

On the other hand, it is impossible to have a boundary Harnack for equations with right hand side in Lipschitz domains with *narrow* corners, even less in Hölder domains or more general domains, under the reasonable hypothesis $\Delta u = f \in L^\infty$, with $\|f\|_{L^\infty}$ small.

However, the boundary Harnack holds for divergence form operators in Lipschitz domains with big Lipschitz constants when the right hand side vanishes as a big enough power of the distance [3]. It is likely that we could prove the same for non-divergence form operators, but we will not pursue this because it cannot be used in the context of free boundary problems.

The following example is based in a counterexample to the Hopf lemma for divergence operators with discontinuous coefficients [155], and shows that the boundary Harnack for equations with a right hand side fails in this setting.

Proposition 4.6.3. *There exists $\mathcal{L}$ in divergence form with discontinuous coefficients and positive functions u, v in $\{y > 0\} \subset \mathbb{R}^2$ that vanish continuously at $\{y = 0\}$ such that*

$$u(1,1) = v(1,1) = 1, \quad \mathcal{L}u = 0 \text{ in } \{y > 0\} \quad \text{and} \quad \|\mathcal{L}v\|_{L^\infty(B_1^+)} < \delta,$$

for any given $\delta > 0$, but

$$\sup_{B_{1/2}^+} \frac{u}{v} = \infty.$$

In particular, Theorem 4.1.2 fails if the divergence form operator has discontinuous coefficients.

Proof. Let $\mathcal{L}u = \mathrm{Div}(A(x,y)\nabla u)$, with

$$A(x,y) = \begin{pmatrix} 1 & -6\,\mathrm{sgn}(x) \\ -6\,\mathrm{sgn}(x) & 48 \end{pmatrix}.$$

It is easy to check that $\mathcal{L}$ is uniformly elliptic and that

$$u(x,y) = \frac{y^3 + 18|x|y^2 + 72x^2y}{91}$$

is a solution of $\mathcal{L}u = 0$. Now we will define v as a perturbation of u, in a similar way as in Proposition 4.6.1. We will use that the coefficients $A(x,y)$ are constant in the positive quadrant.

Let ψ be a positive smooth function such that

$$0 \le \psi \le u, \quad \mathrm{Supp}\,\psi \subset B_{1/3}(1,1) \quad \text{and} \quad \psi(1,1) = u(1,1) = 1.$$

Define the scalings $\psi_\varepsilon(x,y) = \varepsilon^3\psi(x/\varepsilon, y/\varepsilon)$. Since u is homogeneous of degree 3, $0 \le \psi_\varepsilon \le u$. Moreover,

$$\mathcal{L}\psi_\varepsilon(x,y) = \varepsilon(\mathcal{L}\psi)\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right).$$

Now, we construct v as the following infinite sum, that converges uniformly.

$$v := u - \sum_{k=k_0}^{\infty} (1 - 2^{-k})\psi_{2^{-k}}.$$

Since the supports of $\psi_{2^{-k}}$ are disjoint, $v \ge 0$. On the other hand,

$$\|\mathcal{L}v\|_{L^\infty(B_1^+)} \le \sum_{k=k_0}^{\infty} \|\mathcal{L}\psi_{2^{-k}}\|_{L^\infty(B_1^+)} = \sum_{k=k_0}^{\infty} 2^{-k}\|\mathcal{L}\psi\|_{L^\infty(B_1^+)} \to 0$$

as $k_0 \to \infty$. Hence, we can choose k_0 big enough so that $\|\mathcal{L}v\|_{L^\infty(B_1^+)} < \delta$.

To end, for $k \ge k_0$,

$$\frac{u(2^{-k}, 2^{-k})}{v(2^{-k}, 2^{-k})} = 2^k \to \infty,$$

as wanted. □

4.7 Hopf lemma for non-divergence equations with right hand side

We now recall the classical Hopf lemma in a very general version for non-divergence elliptic equations [147].

Theorem 4.7.1. *Suppose that Ω satisfies the interior $\mathcal{C}^{1,\mathrm{Dini}}$ condition at $0 \in \partial\Omega$ and $u \in \mathcal{C}(\overline{\Omega} \cap B_1)$ satisfies*

$$\mathcal{M}^-(D^2u) \le 0 \quad \text{in} \quad \Omega \cap B_1$$

in the L^n-viscosity sense with $u(0) = 0$ and $u \ge 0$ in $\Omega \cap B_1$.

Then for any $l = (l_1, \ldots, l_n) \in \mathbb{R}^n$ with $|l| = 1$ and $l_n > 0$,

$$u(rl) \ge c l_n u(e_n/2) r, \quad r \in (0, \delta),$$

where $c > 0$ and δ depend only on the dimension, λ, Λ and the modulus of continuity of the domain.

We can use this result to prove a generalized Hopf lemma for the solutions of non-divergence equations with small right hand side.

Corollary 4.7.2. *Let $q > n$ and $\mathcal{L}$ in non-divergence form as in (4.2). There exist small $c_0 > 0$ and $L_0 > 0$ such that the following holds.*

Let Ω be a Lipschitz domain as in Definition 4.1.1, with Lipschitz constant $L < L_0$. Suppose further that $\partial\Omega$ is a $\mathcal{C}^{1,\mathrm{Dini}}$ graph. Let v be a solution of

$$\begin{cases} \mathcal{L}v = f & \text{in } \Omega \cap B_1 \\ v = 0 & \text{on } \partial\Omega \cap B_1 \end{cases}$$

in the L^n-viscosity, with $v > 0$ in $\Omega \cap B_1$ and

$$\|f\|_{L^q(B_1)} \le c_0 v(e_n/2).$$

Then, for any $l = (l_1, \ldots, l_n) \in \mathbb{R}^n$ with $|l| = 1$ and $l_n > 0$,

$$v(rl) \ge c l_n v(e_n/2) r, \quad r \in (0, \delta),$$

where c, c_0 and δ are positive and depend only on the dimension, λ, Λ and the modulus of continuity of the domain.

Proof. Assume $v(e_n/2) = 1$ without loss of generality. Let u be a positive solution of the Dirichlet problem

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega \cap B_1 \\ u = 0 & \text{on } \partial\Omega \cap B_1. \end{cases}$$

After dividing by a constant, $u(e_n/2) \le 1$.

Now, by Theorem 4.1.2, we have $u \le Cv$ in $B_{1/2}$, hence the estimate of Theorem 4.7.1 for u is also valid for Cv, and the result follows. □

Chapter 5

Parabolic boundary Harnack inequalities with right-hand side

We prove the parabolic boundary Harnack inequality in parabolic flat Lipschitz domains by blow-up techniques, allowing for the first time a non-zero right-hand side. Our method allows us to treat solutions to equations driven by non-divergence form operators with bounded measurable coefficients, and a right-hand side $f \in L^q$ for $q > n+2$. In the case of the heat equation, we also show the optimal $C^{1-\varepsilon}$ regularity of the quotient.

As a corollary, we obtain a new way to prove that flat Lipschitz free boundaries are $C^{1,\alpha}$ in the parabolic obstacle problem and in the parabolic Signorini problem.

5.1 Introduction

The well known elliptic boundary Harnack inequality asserts that the rate at which positive harmonic functions approach zero Dirichlet boundary conditions depends only on the geometry of the domain. Quantitatively, if u and v are positive harmonic functions in Ω that vanish on $\partial\Omega$, then the quotient u/v is bounded near the boundary. This is also known as the Carleson estimate.

An important corollary of the boundary Harnack is that u/v is not only bounded, but also Hölder continuous. In the elliptic case, the $C^{0,\alpha}$ regularity of the quotient follows from the Carleson estimate by a standard iteration technique (see [47]). However, in the parabolic setting, the question is much more delicate due to the time delay in the interior Harnack inequality. The first proof of the Hölder regularity of the quotient for solutions to the heat equation appeared in [10], two decades after the first proof of the Carleson estimate for caloric functions [126].

When the domains are $C^{1,\mathrm{Dini}}$ or smoother, a combination of the Hopf lemma and the Lipschitz regularity of solutions implies that all solutions to elliptic and parabolic equations decay linearly as they approach zero Dirichlet boundary conditions. However, in less regular domains where the Hopf lemma and Lipschitz continuity do not hold, the fact that the quotient of solutions is bounded is far from trivial and needs to be studied separately.

In this work, we provide a new approach to boundary Harnack inequalities with right-hand side, extending the previous results of Allen and Shahgholian [3], and Ros-Oton and the author [169] to the parabolic setting. Our proof relies on comparison and scaling arguments, as in

[169], as well as a blow-up argument inspired by [167]. We will also consider the applications of our result to free boundary problems, such as the parabolic obstacle problem and the parabolic Signorini problem. Using the boundary Harnack, we can prove $C^{1,\alpha}$ regularity of free boundaries when we know they are flat Lipschitz.

Remark 5.1.1. In flat Lipschitz and C^1 domains, the boundary Harnack and the boundary regularity of solutions do not follow from each other. Nevertheless, they are intimately related, and both can be proved by a contradiction-compactness argument where the Hölder exponent is determined by a Liouville theorem in the half-space.

In this regard, we will prove that, if u and v are positive harmonic functions with zero Dirichlet boundary conditions, $u/v \in C^{1-\varepsilon}$ if the boundary is sufficiently flat, matching the known regularity up to the boundary of u and v, and the fact that the only harmonic function with sublinear growth in a half-space and zero Dirichlet boundary conditions is zero.

5.1.1 Main results

In the following, $\mathcal{L}$ will denote a non-divergence form elliptic operator with bounded measurable coefficients,

$$\mathcal{L}u = \sum_{i,j=1}^{n} a_{ij}(x)\partial^2_{ij}u, \quad \text{with} \quad \lambda I \le A(x) \le \Lambda I, \tag{5.1}$$

with $0 < \lambda \le \Lambda$.

We denote by $\alpha_0(\lambda, \Lambda) \in (0,1)$ a universal constant (only dependent on the dimension and the ellipticity constants), which is defined as the minimum of the following:

- The C^{1,α_0} boundary regularity estimate in [198, Theorem 2.1].
- The interior C^{0,α_0} regularity estimate in [67, Lemma 5.1].

We will define $\alpha_0(1,1) := 1$ instead if the operator is the Laplacian.

Our main result is the following boundary Harnack inequality, which extends the main result in [10] to general non-divergence form operators and equations with right-hand side. Here, $C^{0,\gamma}_p$ is the parabolic Hölder space defined in Section 5.2.

Theorem 5.1.2. *Let $q > n+2$, $0 < \gamma < \min\{\alpha_0, 1 - \frac{n+2}{q}\}$, $m \in (0,1]$, and let $\mathcal{L}$ be a non-divergence form operator as in (5.1). There exists $c_0 \in (0,1)$, only depending on q, γ, the dimension and the ellipticity constants, such that the following holds.*

Let $\Omega \subset \mathbb{R}^{n+1}$ be a parabolic Lipschitz domain in Q_1 in the sense of Definition 5.2.2 with Lipschitz constant $L \le c_0$. Let u and v be solutions to

$$\begin{cases} u_t - \mathcal{L}u &= f_1 \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial_\Gamma \Omega \end{cases} \quad \text{and} \quad \begin{cases} v_t - \mathcal{L}v &= f_2 \quad \text{in} \quad \Omega \\ v &= 0 \quad \text{on} \quad \partial_\Gamma \Omega, \end{cases}$$

and assume that $\|u\|_{L^\infty(Q_1)} \le 1$, $\|v\|_{L^\infty(Q_1)} = 1$, $v > 0$, $v\left(\frac{e_n}{2}, -\frac{3}{4}\right) \ge m$, $\|f_1\|_{L^q(Q_1)} \le 1$ and $\|f_2\|_{L^q(Q_1)} \le c_0 m$.

Then,

$$\left\|\frac{u}{v}\right\|_{C^{0,\gamma}_p(\Omega \cap Q_{1/2})} \le C,$$

where C depends only on q, m, γ, the dimension and the ellipticity constants.

Remark 5.1.3. We will actually prove Theorem 5.1.2 under the more general assumption that $f_i = g_i + h_i$ with

$$\|d^{1-\alpha} g_1\|_{L^\infty(Q_1)} + \|d^{-1/(n+1)-\alpha} h_1\|_{L^{n+1}(Q_1)} \leq 1$$

and

$$\|d^{1-\alpha} g_2\|_{L^\infty(Q_1)} + \|d^{-1/(n+1)-\alpha} h_2\|_{L^{n+1}(Q_1)} \leq c_0 m,$$

where $d(x', x_n, t) = x_n - \Gamma(x', t)$, and $\alpha \in (\gamma, \min\{\alpha_0, 1 - \frac{n+2}{q}\})$ (see Proposition 5.10.2). In this case, C depends also on α.

Remark 5.1.4. The result is sharp in the following sense:

- If the domain is C^1, $\mathcal{L} = \Delta$ and $q = \infty$, we may take $\gamma \to 1$.
- If the Lipschitz constant is not small, the result fails, even for $\mathcal{L} = \Delta$ and $q = \infty$.
- If the norm of the right-hand side is big, the result fails.
- If $q = n + 2$, the result fails for any $c_0 > 0$, even for $\mathcal{L} = \Delta$.

Counterexamples can be constructed by a straightforward adaptation of [169, Section 6] to the parabolic setting. See also [3].

Assuming that both solutions are positive and the right-hand sides of the equations are small, we can use symmetry to deduce the Carleson estimate.

Corollary 5.1.5. *Let $q > n+2$, $m \in (0,1]$, and let $\mathcal{L}$ be a non-divergence form operator as in (5.1). There exists $c_0 \in (0,1)$, only depending on q, the dimension and the ellipticity constants, such that the following holds.*

Let $\Omega \subset \mathbb{R}^{n+1}$ be a parabolic Lipschitz domain in Q_1 in the sense of Definition 5.2.2 with Lipschitz constant $L \leq c_0$. Let u and v be positive solutions to

$$\begin{cases} u_t - \mathcal{L}u = f_1 & \text{in } \Omega \\ u = 0 & \text{on } \partial_\Gamma \Omega \end{cases} \quad \text{and} \quad \begin{cases} v_t - \mathcal{L}v = f_2 & \text{in } \Omega \\ v = 0 & \text{on } \partial_\Gamma \Omega, \end{cases}$$

and assume that $\|u\|_{L^\infty(Q_1)} = \|v\|_{L^\infty(Q_1)} = 1$, $v\left(\frac{e_n}{2}, -\frac{3}{4}\right) \geq m$, $u\left(\frac{e_n}{2}, -\frac{3}{4}\right) \geq m$ and $\|f_i\|_{L^q(Q_1)} \leq c_0 m$.

Then,

$$\frac{1}{C} \leq \frac{u}{v} \leq C \quad \text{in } \Omega \cap Q_{1/2},$$

where C depends only on q, m, the dimension and the ellipticity constants.

We will also deal with solutions to the heat equation in slit domains, that appear naturally when studying the parabolic Signorini problem. Our result gives an alternative proof to the boundary Harnack in [161] when the Lipschitz constant of the domain is small, relaxing the condition on the right-hand side from L^∞ to L^q, and providing the optimal Hölder regularity of the quotient.

Theorem 5.1.6. *Let $q > n+2$, $0 < \gamma < \min\{1, \frac{3}{2} - \frac{n+3}{q}\}$, and $m \in (0,1]$. There exists $c_0 \in (0, \frac{1}{8})$, only depending on q, γ and the dimension such that the following holds.*

Let $\Omega \subset \mathbb{R}^{n+2}$ be a parabolic slit domain in Q_1 in the sense of Definition 5.6.1 with Lipschitz constant $L \le c_0$. Let u and v be solutions to

$$\begin{cases} u_t - \Delta u &= f_1 \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial_\Gamma \Omega \end{cases} \quad \text{and} \quad \begin{cases} v_t - \Delta v &= f_2 \quad \text{in} \quad \Omega \\ v &= 0 \quad \text{on} \quad \partial_\Gamma \Omega, \end{cases}$$

and assume that u and v are even in x_{n+1}, $\|u\|_{L^\infty(Q_1)} \le 1$, $\|v\|_{L^\infty(Q_1)} = 1$, $v > 0$, $v\left(\frac{e_n}{2}, -\frac{3}{4}\right) \ge m$, and that $\|f_1\|_{L^q(Q_1)} \le 1$ and $\|f_2\|_{L^q(Q_1)} \le c_0 m$.

Then,

$$\left\| \frac{u}{v} \right\|_{C_p^{0,\gamma}(\Omega \cap Q_{1/2})} \le C,$$

where C depends only on m, q, γ and the dimension.

Remark 5.1.7. Our method for studying non-divergence form operators relies on constructing homogeneous barriers. While we can create sub- and supersolutions with near-linear homogeneity in almost-flat parabolic Lipschitz domains, this method fails in the case of slit domains when the operator is not the Laplacian. Specifically, there is a gap between the homogeneity of sub- and supersolutions, which is expected when the equation is driven by an operator with coefficients [168, 64]. However, in the case of the Laplacian, we can still construct barriers with homogeneity close to $\frac{1}{2}$, as shown in Proposition 5.6.3.

5.1.2 Known results

The boundary Harnack is a fundamental tool in the realm of analysis and partial differential equations that has had significant impact over the past 50 years. While there is a vast array of literature on this topic and its applications, we have compiled a representative sample of the most noteworthy advancements, exclusively focusing on elliptic and parabolic equations in domains less regular than $C^{1,\mathrm{Dini}}$.

Elliptic boundary Harnack

The first proof of the classical case for harmonic functions in Lipschitz domains was given by Kemper in [125]. Caffarelli, Fabes, Mortola and Salsa considered operators in divergence form in Lipschitz domains, while the case of operators in non-divergence form was treated by Fabes, Garofalo, Marin-Malave and Salsa [41, 84]. Jerison and Kenig extended the same result to NTA domains for divergence form operators [122], and the case of non-divergence operators in Hölder domains with $\alpha > 1/2$ was treated with probabilistic techniques in [22] by Bass and Burdzy. A simple and unified proof of these previous results was recently presented by De Silva and Savin [73, 74].

The boundary Harnack inequality also holds for solutions to elliptic equations with right-hand side. Allen and Shahgholian investigated operators in divergence form in Lipschitz domains with a right-hand side in a weighted L^∞ space [3]. In a subsequent work with Kriventsov, they developed a general theory to derive boundary Harnack inequalities for equations with right-hand side based on the boundary Harnack for homogeneous equations [2]. Ros-Oton and the author studied non-divergence and divergence form operators in Lipschitz domains with small Lipschitz constant and small right-hand side in L^q with $q > n$ [169].

Parabolic boundary Harnack

The presence of a waiting time in the parabolic interior Harnack inequality exacerbates the complexity of the parabolic problem, as it renders several approaches to the elliptic setting inapplicable.

Kemper first established the parabolic boundary Harnack inequality for the heat equation [126]. Fabes, Garofalo, and Salsa extended it to equations in divergence form in Lipschitz cylinders [173, 85]. Athanasopoulos, Caffarelli, and Salsa proved the Hölder continuity of quotients of positive solutions for divergence form operators [10], and Fabes, Safonov, and Yuan extended this result to non-divergence form equations [86, 87]. Bass and Burdzy used probabilistic techniques to handle Hölder cylindrical domains [21], while Hoffman, Lewis, and Nyström treated unbounded parabolically Reifenberg flat domains [117], and Petrosyan and Shi dealt with Lipschitz slit domains, where they allowed for a L^∞ right-hand side in the equation [161]. Recently, De Silva and Savin developed a unified and simplified approach to prove the Carleson estimate (but not the $C^{0,\alpha}$ regularity of the quotient) for both divergence and non-divergence equations in various settings [75].

5.1.3 Parabolic obstacle problems

Boundary Harnack inequalities are a key tool in regularity theory for obstacle problems. They are used to establish $C^{1,\alpha}$ regularity of free boundaries from Lipschitz regularity, following from an original idea of Athanasopoulos and Caffarelli [6]. For an introduction to this strategy, see also [160, Section 6.2] and [97, Section 5.6].

Let us briefly sketch how the technique works. In the elliptic setting, if u is a solution to the obstacle problem

$$\begin{cases} \Delta u &= f\chi_{\{u>0\}} \\ u &\geq 0, \end{cases}$$

then we call $\{u = 0\}$ the contact set and $\partial\{u > 0\}$ the free boundary. The derivatives of u are solutions of

$$\begin{cases} \Delta(\partial_e u) &= \partial_e f \quad \text{in } \{u > 0\} \\ \partial_e u &= 0 \qquad \text{on } \partial\{u > 0\}, \end{cases}$$

where e is an unit vector. Then, applying the boundary Harnack to the derivatives of u yields $u_i/u_n \in C^{0,\alpha}$ for every coordinate i, implying that the normal vector to $\partial\{u > 0\}$ is $C^{0,\alpha}$ and hence $\partial\{u > 0\}$ is $C^{1,\alpha}$.

Previously, this approach was restricted to the case where f was constant because the known boundary Harnack inequalities applied only to equations without a right-hand side. Consequently, alternative methods such as those found in [23, 4, 12] were used to establish C^1 or $C^{1,\alpha}$ regularity of free boundaries where the direct application of boundary Harnack was not feasible. However, new boundary inequalities that accommodate a right-hand side have been developed, enabling simpler proofs and reduced regularity assumptions on the obstacle [3, 169, 2].

In the parabolic obstacle problem with a smooth obstacle, it is well known that the free boundary is $C^{1,\alpha}$ at the points where it is flat Lipschitz [34, 12]. Consider the parabolic obstacle problem

$$\begin{cases} \partial_t u - \Delta u &= f\chi_{\{u>0\}} \\ u &\geq 0. \end{cases} \tag{5.2}$$

Then, we have the following improvement of flatness result for the free boundary. We will state the hypotheses that are typically assumed in regular points as part of the full program to prove free boundary regularity, but we will present a standalone result that is simpler to state and prove.

Corollary 5.1.8. *There exists a dimensional constant $L(n)$ such that the following holds. Let $u \in C_x^{1,1} \cap C_t^1(Q_1)$ be a solution to the parabolic obstacle problem (5.2) with $f \in W^{1,q}$, where $q > n+2$. Assume that $(0,0) \in \partial\{u > 0\}$ and that $\{u > 0\}$ is a parabolic Lipschitz domain in Q_1 in the sense of Definition 5.2.2 with Lipschitz constant $L(n)$. Additionally, assume that $\partial_n u \geq cd$ in $\{u > 0\}$, where $c > 0$ and d is the distance to $\partial\{u > 0\}$.*

Then, $\partial\{u > 0\}$ is a $C^{1,\alpha}$ graph in a neighbourhood of $(0,0)$ for some $\alpha > 0$.

We will now examine the no-sign parabolic obstacle problem, expressed as:

$$\begin{cases} \partial_t u - \Delta u &= f\chi_{\{u \neq 0\}} \\ u &\geq 0, \end{cases} \tag{5.3}$$

This problem was first studied in [44] for $f \equiv 1$. In [4], it was established that f must belong to C^{Dini} to guarantee free boundary regularity, and that the free boundary is C^1 with respect to the parabolic metric at regular points. Furthermore, assuming $f \in W^{1,q}$ with $q > n+2$, we can derive $C^{1,\alpha}$ regularity of the free boundary in space and time at those points using a similar technique to the proof of Corollary 5.1.8.

Our next example is the fully nonlinear parabolic obstacle problem, which has been studied in [12]. The problem is expressed as follows:

$$\begin{cases} \partial_t u - F(D^2 u, x) = f(x)\chi_{\{u>0\}} & \text{in } Q_1 \\ u \geq 0, \quad u_t \geq 0 & \text{in } Q_1, \end{cases} \tag{5.4}$$

where we assume that F satisfies the following conditions:

(H1) F is uniformly elliptic and $F(0, \cdot) \equiv 0$.

(H2) F is convex and C^1 in the first variable.

(H3) F is $W^{1,q}$ in the second variable for some $q > n+2$.

In [12], it is shown that the free boundary is C^∞ at regular points under the assumption that F and f are smooth. Prior studies of the fully nonlinear parabolic obstacle problem in various settings have established the C_p^1 regularity of the free boundary at regular points [106, 120]. We believe that the technique used in [169, Corollary 1.7] for the elliptic case can be adapted to the parabolic problem to derive $C^{1,\alpha}$ regularity of the free boundary when $f \in W^{1,q}$ with $q > n+2$.

5.1.4 The parabolic Signorini problem

The parabolic thin obstacle problem, also known as the parabolic Signorini problem, has been extensively studied in [69]. In $\mathbb{R}^{n+2}$, it can be formulated as follows, with $Q_1^+ := Q_1 \cap \{x_{n+1} > 0\}$:

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } Q_1^+ \\ u \geq 0, \ -\partial_{n+1} u \geq 0, \ u\partial_{n+1} u = 0 & \text{on } \{x_{n+1} = 0\} \end{cases} \tag{5.5}$$

After an even extension in the x_{n+1} variable, solutions to (5.5) also satisfy

$$\begin{cases} \partial_t u - \Delta u = f & \text{in} \quad Q_1 \setminus \Lambda(u) \\ u = 0 & \text{on} \quad \Lambda(u), \end{cases} \tag{5.6}$$

where $\Lambda(u) \subset x_{n+1} = 0$ and $u(x', x_n, x_{n+1}, t) = u(x', x_n, -x_{n+1}, t)$.

Smoothness of the free boundary near regular points has been established in the case where f is smooth [14]. For obstacles with lower regularity, it has been proven in [162] that u_t is continuous at regular free boundary points, implying that the free boundary is locally a $C^{1,\alpha}$ graph in both space and time near regular points, under the assumption that $f \in C^2_p$.

If f is independent of time, corresponding to a stationary obstacle, the arguments presented in [162] hold. In this case, it is expected that the condition for the parabolic Lipschitz regularity of the free boundary, currently established with the assumption that $f \in C^{3/2}_p$ in [69], can be relaxed. On the other hand, once the free boundary is known to be Lipschitz, the weaker assumption that $f \in W^{1,q}$ for some $q > n+2$ is sufficient to deduce that it is $C^{1,\alpha}$, thanks to Theorem 5.1.6. In the following result, we assume the nondegeneracy condition that is expected to hold at regular points, similar to Corollary 5.1.8.

Corollary 5.1.9. *There exists a dimensional constant $L(n)$ such that the following holds. Let $u \in C^{3/2}_x \cap C^1_t(Q_1)$ be a solution to (5.6) with $f \in W^{1,q}$, where $q > n+3$. Assume that $(0,0) \in \partial\{u > 0\}$ and that $\{u > 0\}$ is a parabolic Lipschitz slit domain in Q_1 in the sense of Definition 5.6.1 with Lipschitz constant $L(n)$. Additionally, assume that $\partial_n u \geq c d^{1/2}$ in $\{u > 0\} \cap \{x_{n+1} = 0\}$, where $c > 0$ and d is the distance to $\partial\{u > 0\}$.*

Then, $\partial\{u > 0\}$ is a $C^{1,\alpha}$ graph in a neighbourhood of $(0,0)$ for some $\alpha > 0$ (in the relative topology of $\{x_{n+1} = 0\}$).

Determining the optimal regularity for the obstacle to ensure that the free boundary is $C^{1,\alpha}$ at regular points remains an open problem.

5.1.5 Optimal boundary regularity for the quotient of harmonic functions

We begin by recalling the results for the regularity of harmonic functions in different types of domains (see [97, Section 2.6] and the references therein). If $\Delta u = 0$ in Ω, then the regularity of u depends on the regularity of Ω as follows:

- If Ω is a $C^{1,\alpha}$ domain, $u \in C^{1,\alpha}(\overline{\Omega})$.
- If Ω is a C^1 domain, $u \in C^{1-\varepsilon}(\overline{\Omega})$ for all $\varepsilon > 0$.
- If Ω is a Lipschitz domain with constant L, $u \in C^{0,\gamma}(\overline{\Omega})$, with γ depending only on the dimension and L. Moreover, $\gamma \nearrow 1$ as $L \searrow 0$.

After flattening the boundary with a change of variables, we obtain another angle to see the same phenomenon. If $\mathcal{L}u = 0$ in a half space, where $\mathcal{L}u := \mathrm{Div}(A(x)\nabla u)$ is an elliptic operator in divergence form,

- If $A(x) \in C^{0,\alpha}$, then $u \in C^{1,\alpha}$ by Schauder theory.

- If $A(x) \in C^0$, then $u \in C^{1-\varepsilon}$ for all $\varepsilon > 0$ by Cordes-Nirenberg.
- If $A(x) \in L^\infty$, then $u \in C^{0,\gamma}$ for a small γ by De Giorgi. Moreover, $\gamma \nearrow 1$ as $A(x) \to I$ by Cordes-Nirenberg.

On the other hand, the regularity of the quotient of two harmonic functions with zero Dirichlet boundary data goes as follows. If $\Delta u = \Delta v = 0$ in Ω and $u = v = 0$ on $\partial\Omega$, then

- If Ω is a $C^{1,\alpha}$ domain, $u/v \in C^{1,\alpha}(\overline{\Omega})$, [71].
- If Ω is a C^1 domain, $u/v \in C^{1-\varepsilon}(\overline{\Omega})$ for all $\varepsilon > 0$, [140].
- If Ω is a Lipschitz domain with constant L, $u/v \in C^{0,\gamma}$.

The Hölder exponent in Boundary Harnack inequalities for Lipschitz domains is often suboptimal due to its dependence on an iteration scheme. However, our Theorem 5.1.2 provides a way to bridge the gap between Lipschitz and C^1 domains by showing that as the Lipschitz constant L approaches zero, the exponent γ can approach 1.

Corollary 5.1.10. *Let $\varepsilon > 0$. There exists $L_0 > 0$, only depending on the dimension and ε, such that the following holds.*

Let Ω be a Lipschitz domain in B_1 in the sense of Definition 5.8.1 with Lipschitz constant L_0. Let u and v be solutions to

$$\begin{cases} \Delta u &= 0 \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial_\Gamma \Omega, \end{cases} \qquad \text{and} \qquad \begin{cases} \Delta v &= 0 \quad \text{in} \quad \Omega \\ v &= 0 \quad \text{on} \quad \partial_\Gamma \Omega. \end{cases}$$

Then, $u/v \in C^{1-\varepsilon}(B_{1/2} \cap \overline{\Omega})$.

The proof follows immediately from Theorem 5.1.2.

5.1.6 A general Hopf lemma for parabolic equations with right-hand side

In this section, we present a Hopf lemma for parabolic equations in parabolic $C^{1,\mathrm{Dini}}$ domains. As far as we know, this is the first result of this kind in the parabolic setting that allows for equations with right-hand side. While this is a relatively unexplored area in the literature, similar results have been established previously in the elliptic case [30, 169].

Corollary 5.1.11. *Let $\alpha \in (0, \alpha_0)$, and let $\mathcal{L}$ be a non-divergence form operator as in (5.1). There exists $c_0 > 0$, only depending on α, the dimension, and the ellipticity constants, such that the following holds.*

Let Ω be a parabolic Lipschitz domain in Q_1 in the sense of Definition 5.2.2 with Lipschitz constant $L \le c_0$, and assume that it satisfies the interior $C^{1,\mathrm{Dini}}$ condition at 0 in the sense of Definition 5.9.2. Let u be a positive solution to

$$\begin{cases} u_t - \mathcal{L}u &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial_\Gamma \Omega, \end{cases}$$

and assume that $f = g + h$ *, with*

$$\|d^{1-\alpha} g\|_{L^\infty(Q_1)} + \|d^{-1/(n+1)-\alpha} h\|_{L^{n+1}(Q_1)} \le c_0 u\left(\frac{e_n}{2}, -\frac{1}{2}\right),$$

where $d(x', x_n, t) = x_n - \Gamma(x', t)$*. Then, for all* $r \in (0, \delta)$*,*

$$u(re_n, 0) \ge cr,$$

for some small $c, \delta > 0$*.*

Remark 5.1.12. If the boundary of Ω is a C^1 graph, one can obtain a Lipschitz constant as small as necessary by scaling.

5.1.7 Plan of the paper

The paper is organized as follows.

We begin in Section 5.2 by recalling some classical tools that we will use throughout the paper, such as interior regularity estimates and a Liouville theorem.

In Section 5.3, we derive precise growth estimates near the boundary for solutions of parabolic equations. This allows us to construct a special solution in Section 5.4 that is almost proportional to the distance to the boundary. Using this special solution, we prove our main result, Theorem 5.1.2, in Section 5.5. Our proof relies crucially on the growth properties of the special solution.

In Section 5.6, we apply the same strategy used in Sections 5.3-5.4-5.5 to slit domains and prove Theorem 5.1.6. Furthermore, in Section 5.7, we prove our free boundary regularity results, Corollaries 5.1.8 and 5.1.9.

In Section 5.8, we explain how to apply the ideas of the paper to elliptic equations, leading to the elliptic version of our main theorem, Theorem 5.8.4. Finally, in Section 5.9, we prove Corollary 5.1.11.

5.2 Preliminaries

5.2.1 Setting

Throughout the paper, given $x \in \mathbb{R}^n$, we will denote $x' = (x_1, \ldots, x_{n-1})$. $B_r(x)$ will denote the ball of radius r of $\mathbb{R}^n$, centered at x, and $B'_r(x')$ will be the one of $\mathbb{R}^{n-1}$, with center at x'. We also introduce the parabolic cylinders

$$Q_r(x, t) := B'_r(x') \times (x_n - r, x_n + r) \times (t - r^2, t) \subset \mathbb{R}^{n+1},$$

and we will write $Q_r := Q_r(0, 0)$.

We define the parabolic Lipschitz ($\alpha = 1$) and Hölder ($\alpha \in (0, 1)$) seminorms of a function $g : \Omega \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ as follows:

$$[g]_{C_p^{0,\alpha}(\Omega)} = \sup \frac{|g(x, t) - g(y, s)|}{(|x - y| + |t - s|^{1/2})^\alpha},$$

where the supremum is taken over all pairs of different points $(x,t) \neq (y,s)$ in Ω, and we define correspondingly the parabolic Lipschitz and Hölder norms

$$\|g\|_{C_p^{0,\alpha}(\Omega)} := \|g\|_{L^\infty(\Omega)} + [g]_{C_p^{0,\alpha}(\Omega)}.$$

We will omit the domain where there is no case for confusion.

We will denote by

$$\mathcal{M}^-(D^2u) := \inf_{\lambda I \leq A \leq \Lambda I} \mathrm{Tr}(AD^2u), \quad \mathcal{M}^+(D^2u) := \sup_{\lambda I \leq A \leq \Lambda I} \mathrm{Tr}(AD^2u)$$

the Pucci extremal operators; see [37] or [97] for their properties.

In our work, we will consider the following notion of solutions.

Definition 5.2.1. Let $f \in L^{n+1}_{\mathrm{loc}}$. We say $u \in C^0 \cap L^{n+1}_{\mathrm{loc}}$ is a *strong* solution to

$$u_t - \mathcal{L}u = f$$

if $D^2_x u, \partial_t u \in L^{n+1}_{\mathrm{loc}}$, and the equation holds almost everywhere. The condition $u \in C^0$ is redundant, but we write it to fix ideas, see [24, Theorem 10.4].

We will consider parabolic Lipschitz domains of the following form.

Definition 5.2.2. We say Ω is a parabolic Lipschitz domain in Q_R with Lipschitz constant L if Ω is the epigraph of a parabolic Lipschitz function $\Gamma : B'_R \times [-R^2, 0] \to \mathbb{R}$, with $\Gamma(0,0) = 0$:

$$\Omega = \Big\{(x,t) \in Q_R \mid x_n > \Gamma(x',t)\Big\}, \quad \|\Gamma\|_{C_p^{0,1}} \leq L.$$

In this context, we will denote the lateral boundary

$$\partial_\Gamma \Omega := \Big\{(x,t) \in Q_R \mid x_n = \Gamma(x',t)\Big\},$$

and the parabolic boundary

$$\partial_p \Omega := \partial_\Gamma \Omega \cup \Big(\overline{\Omega} \cap \partial Q_R \cap \{t < 0\}\Big).$$

5.2.2 Technical tools

Let us start with the parabolic interior Harnack.

Theorem 5.2.3 ([197, Theorem 4.18])**.** *Let $\mathcal{L}$ be a non-divergence form operator as in (5.1), and let u be a strong solution to $u_t - \mathcal{L}u = 0$ in Q_r, with $u \geq 0$.*

Then,

$$\sup_{Q_{r/2}\left(0,-\frac{r}{2}\right)} u \leq C \inf_{Q_{r/2}} u,$$

where C depends only on the dimension and the ellipticity constants.

Then, we recall the Alexandrov-Bakelman-Pucci-Krylov-Tso estimate; see [138, 195].

Theorem 5.2.4. *Let $\mathcal{L}$ be a non-divergence form operator as in (5.1) and let u be a strong solution to $u_t - \mathcal{L}u = f$ in Q_r, with $f \in L^{n+1}(Q_r)$.*

Then,

$$\sup_{Q_r} u \leq \sup_{\partial_p Q_r} u^+ + Cr^{n/(n+1)} \|f\|_{L^{n+1}(Q_r)},$$

where C depends only on the dimension and the ellipticity constants.

We will state together the interior regularity estimates for the heat equation and for equations with bounded measurable coefficients.

Theorem 5.2.5. *Let $\mathcal{L}$ be a non-divergence form operator as in (5.1), let $\alpha \in (0, \alpha_0)$, and let u be a strong solution to $u_t - \mathcal{L}u = f$ in Q_r, with $f \in L^{n+1}(Q_r)$.*

Then,

$$[u]_{C_p^{0,\alpha}(Q_{r/2})} \leq C(r^{-\alpha}\|u\|_{L^\infty(Q_r)} + r^{n/(n+1)-\alpha}\|f\|_{L^{n+1}(Q_r)}),$$

where C depends only on α, the dimension and the ellipticity constants.

Proof. Let us write the proof for $r = 1$, as the dependence on r follows by scaling.

First, when $\mathcal{L}$ is the Laplacian, we apply the Calderón-Zygmund estimates in [199, Theorem 6] to obtain:

$$\|D^2u\|_{L^{n+1}(Q_{1/2})} + \|u_t\|_{L^{n+1}(Q_{1/2})} \leq C(\|u\|_{L^\infty(Q_1)} + \|f\|_{L^{n+1}(Q_1)}).$$

The conclusion follows via the parabolic Sobolev embedding in [201, Theorem 1.4.1 (ii)].

In the case of operators with coefficients, this result is [67, Lemma 5.1].[1] □

We also need the following covering result (cf. [140, Lemma B.2]).

Lemma 5.2.6. *Let $\alpha \in (0, 1)$, and let Ω be a parabolic Lipschitz domain in Q_1 with Lipschitz constant $\frac{1}{8}$, in the sense of Definition 5.2.2.*

Assume that $u : \Omega \cap Q_1 \to \mathbb{R}$ satisfies

$$[u]_{C_p^{0,\alpha}(Q_r(x_0,t_0))} \leq C_0$$

whenever $Q_{2r}(x_0, t_0) \subset \Omega \cap Q_1$. Then, for any $0 < \sigma < 1$,

$$[u]_{C_p^{0,\alpha}(\Omega \cap Q_\sigma)} \leq CC_0.$$

The constant C depends only on α and σ.

The proof follows by a standard covering argument (see [97, Appendix A]). In addition, the following convergence result for limits of solutions will be useful to make contradiction-compactness arguments.

Proposition 5.2.7 ([67, Theorem 6.1]). *Let $f_k \in L^{n+1}(\Omega)$ and let u_k be solutions to*

$$(\partial_t - \mathcal{M}^+)u_k \leq f_k \leq (\partial_t - \mathcal{M}^-)u_k \quad \text{in } \Omega,$$

such that $u_k \to u_0$ locally uniformly, and that for every $K \subset\subset \Omega$, $\|f_k\|_{L^{n+1}(K)} \to 0$.

Then,

$$(\partial_t - \mathcal{M}^+)u_0 \leq 0 \leq (\partial_t - \mathcal{M}^-)u_0$$

in the viscosity sense.

[1] The result is actually stated for L^p-viscosity solutions, but it automatically extends to strong solutions. See the introduction of [67].

Finally, a key step in our proofs is to classify certain solutions in special domains. We will do so with the following Liouville theorem for the half-space, that follows from standard arguments from the boundary regularity estimate [198, Theorem 2.1].

Theorem 5.2.8. *Let $\mathcal{L}$ be a non-divergence form operator as in (5.1), let $\alpha \in (0, \alpha_0)$ and let u be a solution to*

$$\begin{cases} (\partial_t - \mathcal{M}^+)u \leq 0 \leq (\partial_t - \mathcal{M}^-)u & \text{in } \{x_n > 0\} \\ u = 0 & \text{in } \{x_n \leq 0\} \end{cases}$$

with the growth condition

$$\|u\|_{L^\infty(Q_R)} \leq C(1 + R^{1+\alpha}), \ \forall R \geq 1.$$

Then, $u = k(x_n)_+$ for some $k \in \mathbb{R}$.

Proof. First, observe that, for every $r \geq 1$,

$$u_r(x,t) := \frac{u(rx, r^2 t)}{r^{1+\alpha}}$$

also satisfies the hypotheses.

Now, from [198, Theorem 2.1] and interior regularity estimates (in the case of the heat equation, by C^2 boundary regularity estimates) it follows that

$$[u]_{C^{1,\alpha_0}(Q_r)} = r^{\alpha-\alpha_0}[u_r]_{C^{1,\alpha_0}(Q_1)} \leq C r^{\alpha-\alpha_0}\|u_r\|_{L^\infty(Q_2)} \leq C r^{\alpha-\alpha_0},$$

and then letting $r \to \infty$ we deduce that

$$[u]_{C^{1,\alpha_0}(\mathbb{R}^{n+1})} = 0,$$

and therefore u is a linear function. From the boundary conditions, we deduce that $u = k(x_n)_+$. □

5.3 Boundary growth and regularity estimates

The main goal of this section is to prove that for sufficiently flat domains, solutions that vanish on the boundary of the domain grow like $d^{1\pm\varepsilon}$ and are $C_p^{0,\gamma}$.

Proposition 5.3.1. *Let $\gamma \in (0, \alpha_0)$. There exists $L_0 > 0$, depending only on γ, the dimension and the ellipticity constants, such that the following holds.*

Let Ω be a parabolic Lipschitz domain in Q_1 as in Definition 5.2.2 with Lipschitz constant $L \leq L_0$. Let $d(x', x_n, t) = x_n - \Gamma(x', t)$, and let $u \in C(Q_1)$ be a viscosity solution to

$$u_t - \mathcal{M}^- u \geq -K_0(d^{\gamma-2} + f) \quad \text{and} \quad u_t - \mathcal{M}^+ u \leq K_0(d^{\gamma-2} + f) \quad \text{in } \Omega \cap Q_1,$$

$$u = 0 \quad \text{in } \partial_\Gamma \Omega \cap Q_1.$$

Assume that $\|d^{-(\gamma - n/(n+1))_+} f\|_{L^{n+1}(Q_1)} \leq 1$. Then,

$$\|u\|_{C_p^{0,\gamma}(Q_{1/2})} \leq C(\|u\|_{L^\infty(Q_1)} + K_0).$$

The constant C depends only on n, γ and the ellipticity constants.

We start by introducing the regularized distance [148, 149], a technical tool that will be useful to construct barriers.

Lemma 5.3.2. *Let Ω be a parabolic Lipschitz domain with Lipschitz constant $L \leq 1$, in the sense of Definition 5.2.2. Then, there exists a function $d : \Omega \cap Q_1 \to \mathbb{R}$ satisfying the following:*

$$\frac{1}{2}(x_n - \Gamma(x', t)) \leq d \leq \frac{3}{2}(x_n - \Gamma(x', t))$$
$$\frac{2}{3} \leq |\nabla_x d| \leq C_1$$
$$|\partial_t d| + |D_x^2 d| \leq C_2 L d^{-1},$$

where C_1 and C_2 depend only on the dimension.

We will sketch the proof in Appendix 5.10.

As a direct consequence, we obtain the following scaling property.

Lemma 5.3.3. *Let Ω be a parabolic Lipschitz domain in Q_R, in the sense of Definition 5.2.2. Let $r \in (0, 1)$ and let*

$$\tilde{\Omega} := \{(x, t) \in Q_R \mid (rx, r^2 t) \in \Omega\}.$$

Let d and $\tilde{d}$ be the regularized distances in Ω and $\tilde{\Omega}$, respectively. Then,

$$\frac{1}{3}\tilde{d}(x, t) \leq \frac{1}{r} d(rx, r^2 t) \leq 3\tilde{d}(x, t).$$

Proof. It follows from Lemma 5.3.2. □

The following barriers are constructed in a very similar way to [140, Lemmas 3.2 and 3.3]. We start with a supersolution.

Lemma 5.3.4. *Let $\varepsilon \in (0, 1)$. There exist sufficiently small $\eta > 0$ and sufficiently large $K > 0$, only depending on the dimension, ε and the ellipticity constants, such that the following holds.*

Let u be a solution to

$$\begin{cases} u_t - \mathcal{M}^+ u & \leq & \eta d^{-1-\varepsilon} & \text{in} & \Omega \\ u & \leq & 1 & \text{on} & \partial_p \Omega \\ u & \leq & 0 & \text{on} & \partial_\Gamma \Omega, \end{cases}$$

where Ω is a parabolic Lipschitz domain in Q_1 in the sense of Definition 5.2.2 with Lipschitz constant η. Then,

$$u \leq K d^{1-\varepsilon} - t + |x'|^2 \quad \text{in } \Omega,$$

where d is the regularized distance introduced in Lemma 5.3.2, and

$$u(re_n, 0) \leq K r^{1-\varepsilon}, \quad \forall r \in (0, 1).$$

Proof. We will use the comparison principle with a supersolution that has the desired growth.

Let

$$\varphi = K d^{1-\varepsilon} - t + |x'|^2,$$

where d is the regularized distance introduced in Lemma 5.3.2, and $K > 0$ is a large constant to be chosen later.

Notice that $\varphi \geq 0$ on $\partial_\Gamma \Omega$, and that $\varphi \geq 1$ on $\{t=-1\}$ and $\{|x'|=1\}$. We can also check

$$\varphi \geq K\left(\frac{x_n - \Gamma(x',t)}{2}\right)^{1-\varepsilon} - t + |x'|^2 > \frac{K(1-2\eta)}{2} \geq 1 \quad \text{on} \quad \{x_n = 1\} \cap \overline{\Omega}.$$

On the other hand, using the estimates in Lemma 5.3.2,

$$\begin{aligned}
(\partial_t - \mathcal{M}^+) d^{1-\varepsilon} &= (1-\varepsilon) d^{-\varepsilon}\partial_t d - \sup_{\lambda I \leq A \leq \Lambda I} \mathrm{Tr}(A D_x^2 d^{1-\varepsilon}) \\
&= (1-\varepsilon)\left(d^{-\varepsilon}\partial_t d - \sup_{\lambda I \leq A \leq \Lambda I}\left[\sum_{i=1}^n \sum_{j=1}^n a_{ij}(d^{-\varepsilon}\partial_{ij}^2 d - \varepsilon d^{-1-\varepsilon}\partial_i d \partial_j d)\right]\right) \\
&= (1-\varepsilon) d^{-\varepsilon}(\partial_t - \mathcal{M}^+) d + \varepsilon(1-\varepsilon) d^{-1-\varepsilon} \inf_{\lambda I \leq A \leq \Lambda I} \nabla_x d^\top A \nabla_x d \\
&\geq (1-\varepsilon)\left(-C\eta + \frac{4\lambda\varepsilon}{9}\right) d^{-1-\varepsilon},
\end{aligned}$$

where we omitted the dependence of A and a_{ij} on x for formatting. Then,

$$(\partial_t - \mathcal{M}^+)\varphi \geq K(1-\varepsilon)\left(-C\eta + \frac{4\lambda\varepsilon}{9}\right) d^{-1-\varepsilon} - 1 - 2(n-1)\Lambda \geq \eta d^{-1-\varepsilon},$$

where we chose η small, K large, and used that $d < \frac{3}{2}$.

Therefore, applying the comparison principle to u and φ, we obtain that $u \leq \varphi$ in Ω, and in particular

$$u(re_n, 0) \leq \varphi(re_n, 0) = Kd(re_n, 0)^{1-\varepsilon} \leq K' r^{1-\varepsilon}.$$

□

Analogously, we can consider a subsolution.

Lemma 5.3.5. *Let $\varepsilon \in (0,1)$. There exist sufficiently small $\eta, k, r_0 > 0$, only depending on the dimension, ε and the ellipticity constants, such that the following holds.*

Let $u \geq 0$ be a solution to

$$\begin{cases} u_t - \mathcal{M}^- u \geq -\eta d^{\varepsilon - 1} & \text{in} \quad \Omega \\ \qquad\qquad u \geq 1 & \text{on} \quad \partial_{\mathrm{up}}\Omega \end{cases}$$

where Ω is a parabolic Lipschitz domain in Q_1 in the sense of Definition 5.2.2 with Lipschitz constant η, and

$$\partial_{\mathrm{up}}\Omega := \{x_n = 1\} \cap \overline{\Omega}$$

is the top part of the boundary. Then,

$$u \geq kd^{1+\varepsilon} + t - |x'|^2 \quad \text{in } \Omega \cap Q_{r_0},$$

where d is the regularized distance introduced in Lemma 5.3.2, and

$$u(re_n, 0) \geq kr^{1+\varepsilon}, \quad \forall r \in (0,1).$$

Proof. First, let

$$\Omega^{(1)} := \left\{(x,t) \in \Omega : |x'| < \frac{1}{2}, x_n < r_0, -\frac{1}{2} < t\right\},$$
$$\Omega^{(2)} := \left\{(x,t) \in \Omega : |x'| < \frac{1}{2}, x_n > r_0, -\frac{1}{2} < t\right\},$$

and

$$\Omega^{(3)} := Q_1 \setminus \left\{x_n \le \frac{r_0}{2}\right\},$$

with $r_0 \in (0,1)$ to be chosen later. Let also

$$\partial_{\text{up}}\Omega^{(1)} := \{x_n = r_0\} \cap \overline{\Omega^{(1)}}.$$

Now, let v be the solution to

$$\begin{cases} v_t - \mathcal{M}^- v &= 0 \quad \text{in} \quad \Omega^{(3)} \\ v &= 1 \quad \text{on} \quad \partial_{\text{up}}\Omega \\ v &= 0 \quad \text{in} \quad \partial_p\Omega^{(3)} \setminus \partial_{\text{up}}\Omega. \end{cases}$$

By the strong maximum principle, $\min_{\overline{\Omega^{(2)}}} v \ge 2c_0 > 0$. On the other hand, by the comparison principle,

$$w := u + 2\eta r_0^{\varepsilon-1}(t+1) \ge v,$$

because $w \ge u$ on $\partial_p\Omega^{(3)}$ and

$$(\partial_t - \mathcal{M}^-)w = (\partial_t - \mathcal{M}^-)u + 2\eta r_0^{\varepsilon-1} \ge \eta(-d^{\varepsilon-1} + 2r_0^{\varepsilon-1}) \ge 0.$$

Hence, noting that $\partial_{\text{up}}\Omega^{(1)} \subset \overline{\Omega^{(2)}}$,

$$\min_{\partial_{\text{up}}\Omega^{(1)}} u \ge \min_{\overline{\Omega^{(2)}}} v - 2r_0^{\varepsilon-1}\eta \ge c_0 > 0,$$

and $\min_{r\in[r_0,1]} u(re_n, 0) \ge c_0$ as well, choosing η small enough.

Now, let

$$\varphi(x,t) = kd^{1+\varepsilon} + t - |x'|^2,$$

where d is the regularized distance introduced in Lemma 5.3.2, for the domain Ω (that coincides with the regularized distance for the domain $\Omega^{(1)}$ because the two domains lie above the same graph), and $k = \min\{c_0/(4r_0), 1/32\}$.

Then, $\varphi \le u$ on the parabolic boundary of $\Omega^{(1)}$. Indeed, since $d \le 2r_0$ on $\partial_{\text{up}}\Omega^{(1)}$, $\varphi < c_0$ on $\partial_{\text{up}}\Omega^{(1)}$. When $t = -1/2$, since $d \le 2$,

$$\varphi \le \frac{1}{32}d^{1+\varepsilon} - \frac{1}{2} \le \frac{1}{8} - \frac{1}{2} < 0,$$

and if $|x'| = 1/2$,

$$\varphi \le \frac{1}{8} - \frac{1}{4} < 0.$$

Finally, $\varphi \le 0$ on $\partial_\Gamma\Omega$.

Hence, by an analogous computation to the proof of Lemma 5.3.4, we obtain

$$\varphi_t - \mathcal{M}^-\varphi \le (1+\varepsilon)k\left(C\eta - \frac{4\lambda\varepsilon}{9}\right)d^{\varepsilon-1} + 1 + 2(n-1)\Lambda \le -\eta d^{\varepsilon-1},$$

using that η and $d < 2r_0$ can be chosen arbitrarily small.

Therefore, applying the comparison principle to u and φ, we obtain that $u \ge \varphi$ in $\Omega^{(1)}$, and in particular in $\Omega \cap Q_{r_0}$, and also that for all $r \in (0, r_0)$

$$u(re_n, 0) \ge kd^{1+\varepsilon} \ge k'r^{1+\varepsilon}.$$

On the other hand, if $r > r_0$, we can use directly that $u(re_n, 0) \ge c_0$. □

The growth upper bound still holds when we consider a (small) right-hand side in L^{n+1}, thanks to Theorem 5.2.4.

Lemma 5.3.6. *Let $\varepsilon \in (0,1)$. There exist sufficiently small $\eta > 0$ and sufficiently large $K > 0$, only depending on the dimension, ε, and the ellipticity constants, such that the following holds.*

Let $d(x', x_n, t) = x_n - \Gamma(x', t)$, and let u be a solution to

$$\begin{cases} u_t - \mathcal{M}^+ u & \le & \eta d^{-1-\varepsilon} + f & \text{in} & \Omega \\ u & \le & 1 & \text{on} & \partial_p\Omega \\ u & \le & 0 & \text{on} & \partial_\Gamma\Omega \end{cases}$$

where $\|d^{-(1/(n+1)-\varepsilon)_+}f\|_{L^{n+1}(\Omega)} \le \eta$, and Ω is a parabolic Lipschitz domain in Q_1 in the sense of Definition 5.2.2 with Lipschitz constant η. Then,

$$u(re_n, 0) \le Kr^{1-\varepsilon}, \quad \forall r \in (0,1).$$

Remark 5.3.7. Thanks to Lemma 5.3.2, we can interchange the regularized distance with $x_n - \Gamma(x', t)$ up to a constant. We will do so in the following.

Proof. We will iterate Lemma 5.3.4 combined with Theorem 5.2.4. We define the rescaled functions

$$u_j(x,t) := \frac{u(\rho^j x, \rho^{2j}t)}{\rho^{j(1-\varepsilon)}},$$

with $\rho > 0$ to be chosen later.

Now,

$$(\partial_t - \mathcal{M}^+)u_j \le \rho^{j(1+\varepsilon)}(\eta(\rho^j d)^{-1-\varepsilon} + \tilde{f}_j) \le \eta d^{-1-\varepsilon} + \rho^{j(1+\varepsilon)}\tilde{f}_j,$$

with $\tilde{f}_j(x,t) := f(\rho^j x, \rho^{2j}t)$.

Let $a_j := \|u_j\|_{L^\infty(\tilde{\Omega})}$. By Lemma 5.3.4 and Theorem 5.2.4, $a_0 \le C_0$. We will show by induction that $a_j \le C_0$ for all $j \ge 0$.

Again by Lemma 5.3.4 (with $\varepsilon/2$) and Theorem 5.2.4 applied to u_j,

$$\begin{aligned} a_{j+1}\rho^{1-\varepsilon} &\le K\rho^{1-\varepsilon/2}a_j + 2\rho^2 a_j + C\rho^{j(1+\varepsilon)}\|\tilde{f}_j\|_{L^{n+1}(\tilde{\Omega})} \\ &\le K\rho^{1-\varepsilon/2}C_0 + 2\rho^2 C_0 + C\rho^{j(\varepsilon-1/(n+1))}\|f\|_{L^{n+1}(\rho^j\Omega)} \\ &\le \rho^{1-\varepsilon}C_0/2 + 2\rho^2 C_0 + C\|d^{-(1/(n+1)-\varepsilon)_+}f\|_{L^{n+1}(\Omega)} \le \rho^{1-\varepsilon}C_0, \end{aligned}$$

choosing adequately small ρ and η.

The conclusion follows by observing that given $r \in (0,1)$, for all j such that $\rho^j \ge r$, $u(re_n, 0) \le a_j\rho^{j(1-\varepsilon)}$, and hence $u(re_n, 0) \le C_0(r/\rho)^{1-\varepsilon}$. □

Conversely, we can also add a more general right-hand side to the equation for the subsolution.

Lemma 5.3.8. *Let $\varepsilon \in (0,1)$. There exist sufficiently small $\eta, k > 0$, only depending on the dimension, ε, and the ellipticity constants, such that the following holds.*

Let $d(x', x_n, t) = x_n - \Gamma(x', t)$, and let $u \geq 0$ be a solution to

$$\begin{cases} u_t - \mathcal{M}^- u & \geq & -\eta d^{\varepsilon-1} + f & \text{in} & \Omega \\ u & \geq & 1 & \text{on} & \partial_{\text{up}}\Omega \end{cases}$$

where $\|d^{-\varepsilon-1/(n+1)} f\|_{L^{n+1}(\Omega)} \leq \eta$, and Ω and $\partial_{\text{up}}\Omega$ are defined as in Lemma 5.3.5. Then,

$$u(re_n, 0) \geq kr^{1+\varepsilon}, \quad \forall r \in (0,1).$$

Proof. We will use a similar strategy to the proof of Lemma 5.3.6. We define the rescaled functions

$$u_j(x,t) := \frac{u(\rho^j x, \rho^{2j} t)}{\rho^{j(1+\varepsilon)}},$$

with $\rho > 0$ to be chosen later.

Now,

$$(\partial_t - \mathcal{M}^-) u_j \geq \rho^{j(1-\varepsilon)}(-\eta(\rho^j d)^{-1+\varepsilon} + \tilde{f}_j) \geq -\eta d^{-1+\varepsilon} + \rho^{j(1-\varepsilon)} \tilde{f}_j,$$

with $\tilde{f}_j(x,t) := f(\rho^j x, \rho^{2j} t)$.

Let $a_j := \inf_{\partial_{\text{up}}\Omega} u_j$. By hypothesis, $a_0 = 1$. We will show by induction that $a_j \geq 1$ for all $j \geq 0$.

Again by Lemma 5.3.5 (with $\varepsilon/2$) and Theorem 5.2.4, applied to u_j,

$$\begin{aligned} a_{j+1}\rho^{1+\varepsilon} &\geq k\rho^{1+\varepsilon/2} a_j - 2\rho^2 - C\rho^{j(1-\varepsilon)} \|\tilde{f}_j\|_{L^{n+1}(\tilde{\Omega})} \\ &\geq k\rho^{1+\varepsilon/2} - 2\rho^2 - C\rho^{-j(\varepsilon+1/(n+1))} \|f\|_{L^{n+1}(\rho^j\Omega)} \\ &\geq 2\rho^{1+\varepsilon} - 2\rho^2 - C\|d^{-\varepsilon-1/(n+1)} f\|_{L^{n+1}(\Omega)} \geq \rho^{1+\varepsilon}, \end{aligned}$$

choosing adequately small ρ and η.

Finally, by Lemma 5.3.5 and Theorem 5.2.4,

$$u_j(re_n, 0) \geq ka_j - C\eta \geq k' > 0,$$

for all $r \in (\rho, 1)$, provided that η is small enough. Hence, undoing the scaling, $u(re_n, 0) \geq k' r^{1+\varepsilon}$, as we wanted to prove. □

Combining the previous estimates, we can now give the following.

Proposition 5.3.9. *Let $\varepsilon \in (0,1)$ and let $\mathcal{L}$ be a non-divergence form operator as in (5.1). There exists sufficiently small $\eta > 0$, only depending on the dimension, ε and the ellipticity constants, such that the following holds.*

Let Ω be a parabolic Lipschitz domain in Q_1 with Lipschitz constant η in the sense of Definition 5.2.2. Let $d(x', x_n, t) = x_n - \Gamma(x', t)$, and let u be a solution to

$$\begin{cases} u_t - \mathcal{L}u & = & f & \text{in} & \Omega, \\ u & = & 0 & \text{on} & \partial_\Gamma\Omega. \end{cases}$$

Assume that $\|u\|_{L^\infty(Q_1)} \le 1$, *and* $f = g + h$, *with*

$$\|d^{\varepsilon+1} g\|_{L^\infty(Q_1)} + \|d^{-(1/(n+1)-\varepsilon)_+} h\|_{L^{n+1}(Q_1)} \le \eta.$$

Then,

$$|u| \le C d^{1-\varepsilon} \quad \textit{in} \quad \Omega \cap Q_{3/4}.$$

Moreover, if u *is nonnegative,* $m = u\left(\frac{e_n}{2}, -\frac{3}{4}\right) > 0$ *and*

$$\|d^{-\varepsilon+1} g\|_{L^\infty(Q_1)} + \|d^{-\varepsilon-1/(n+1)} h\|_{L^{n+1}(Q_1)} \le \eta m,$$

then,

$$u \ge c m d^{1+\varepsilon} \quad \textit{in} \quad \Omega \cap Q_{3/4}.$$

The constants C *and* c *are positive and depend only on the dimension,* ε*, and the ellipticity constants.*

Proof. For the first estimate, let $(x_0, t_0) \in \partial_\Gamma \Omega \cap Q_{3/4}$, and consider the function

$$v(x,t) := u\left(x_0 + \frac{1}{4}x, t_0 + \frac{1}{16}t\right).$$

Then, by Lemma 5.3.6, $v(re_n, 0) \le K r^{1-\varepsilon}$ for all $r \in (0,1)$. Since d is comparable to $x_n - \Gamma(x', t)$, it follows that $u \le C d^{1-\varepsilon}$ in $\Omega \cap Q_{3/4} \cap \{x_n < 1/4\}$.

Finally, notice that

$$d \ge \frac{1}{2}(x_n - \Gamma(x', t)) \ge \frac{1}{2}\left(\frac{1}{4} - 2\eta\right) > \frac{1}{9} \quad \text{in } \Omega \cap \{x_n \ge 1/4\},$$

and the conclusion follows adjusting C if necessary.

For the second estimate, let $\Omega(x_0, t_0) := \Omega \cap Q_{1/8}(x_0, t_0)$ and notice that

$$\bigcup_{(x_0,t_0) \in \partial_\Gamma \Omega \cap Q_{3/4}} \partial_{\mathrm{up}} \Omega(x_0, t_0) \subset E := \overline{B'_{7/8}} \times [1/8 - 2\eta, 1/8 + 2\eta] \times [-37/64, 0],$$

where $\partial_{\mathrm{up}} \Omega(x_0, t_0) := \overline{\Omega(x_0, t_0)} \cap \{x_n = x_{0,n} + 1/8\}$, analogously to Lemma 5.3.5.

Then, by the interior Harnack (Theorem 5.2.3), $u \ge c_1 m$ in E, and by an analogous reasoning to the upper bound with Lemma 5.3.8 instead of Lemma 5.3.6, the conclusion follows. □

We are finally able to prove our $C^{0,\gamma}$ boundary regularity result.

Proof of Proposition 5.3.1. We may assume that $\|u\|_{L^\infty(Q_1)} \le 1$ and $K_0 = \eta$ (with η from Proposition 5.3.9) without loss of generality after dividing by a constant. Then, by Proposition 5.3.9 and Lemma 5.3.2,

$$|u| \le K((x_n - \Gamma(x', t))^\gamma \quad \text{in } \Omega \cap Q_{3/4}.$$

Then, we will use interior estimates in combination with Lemma 5.2.6 to deduce the result.

Let $p = (y', y_n, s)$ and $\rho \in (0, \frac{1}{16})$ such that $Q_{2\rho}(p) \subset \Omega \cap Q_{5/8}$, and let

$$R := \max\left\{\rho, \frac{y_n - \Gamma(y', s)}{3}\right\}.$$

Note that $Q_{2R}(p) \subset \Omega$. We distinguish two cases:

Case 1. $R \geq \frac{1}{16}$. Then, $Q_{1/8}(p) \subset \Omega \cap Q_{3/4}$, and for all $(x', x_n, t) \in Q_{1/8}(p)$,

$$x_n \geq \Gamma(x', t) + \frac{R}{2} \geq \Gamma(x', t) + \frac{1}{32}.$$

Hence,

$$\begin{aligned} u_t - \mathcal{M}^- u &\geq -d^{\gamma-2} + f \geq -2^{12} + f, \\ u_t - \mathcal{M}^+ u &\leq d^{\gamma-2} + f \leq 2^{12} + f, \end{aligned}$$

which together with the fact that $\|u\|_{L^\infty(Q_1)} \leq 1$, Theorem 5.2.5, and a covering argument, gives

$$[u]_{C_p^{0,\gamma}(Q_\rho(p))} \leq [u]_{C_p^{0,\gamma}(Q_{1/16}(p))} \leq C.$$

Case 2. $R < \frac{1}{16}$. Notice that if $\rho < R$, $y_n - \Gamma(y', s) = 3R$, and if $\rho = R$, using that $Q_{2\rho}(p) \subset \Omega$, $y_n - \Gamma(y', s) \geq 2\rho = 2R$. In either case,

$$2R \leq y_n - \Gamma(y', s) \leq 3R.$$

Now, for all $(x', x_n, t) \in Q_{3R/2}(p)$,

$$\begin{aligned} x_n - \Gamma(x', t) &\geq y_n - \Gamma(y', s) - \frac{3}{2}R - |\Gamma(x', t) - \Gamma(y', s)| \geq \frac{R}{4}, \\ x_n - \Gamma(x', t) &\leq y_n - \Gamma(y', s) + \frac{3}{2}R + |\Gamma(x', t) - \Gamma(y', s)| \leq 5R \end{aligned}$$

using the parabolic Lipschitz character of Γ and that $L_0 \leq \frac{1}{8}$.

Therefore,

$$\begin{aligned} u_t - \mathcal{M}^- u &\geq -CR^{\gamma-2} + f, \\ u_t - \mathcal{M}^+ u &\leq CR^{\gamma-2} + f, \end{aligned}$$

and $\|u\|_{L^\infty(Q_{3R/2})} \leq K(5R)^\gamma$, which combined with Theorem 5.2.5 gives

$$\begin{aligned} [u]_{C_p^{0,\gamma}(Q_R(p))} &\lesssim R^{-\gamma}(5R)^\gamma + R^{n/(n+1)-\gamma}\|R^{\gamma-2} + f\|_{L^{n+1}(Q_{3R/2}(p))} \\ &\lesssim 1 + R^{n/(n+1)-\gamma}|Q_{3R/2}|^{1/(n+1)}R^{\gamma-2} + R^{n/(n+1)-\gamma}\|f\|_{L^{n+1}(Q_{3R/2}(p))} \\ &\lesssim 1 + 1 + \|d^{-(\gamma-n/(n+1))_+} f\|_{L^{n+1}(Q_{3R/2}(p))} \lesssim 1. \end{aligned}$$

The conclusion follows by Lemma 5.2.6. □

5.4 The near-linear solution

Our goal now is to find a special solution satisfying the following.

Proposition 5.4.1. *Let $\varepsilon \in (0, \alpha_0)$, and let $\mathcal{L}$ be a non-divergence form operator as in (5.1). Then, there exists $\delta > 0$, only depending on ε, the dimension and the ellipticity constants, such that the following holds.*

Let Ω be a parabolic Lipschitz domain in Q_1 in the sense of Definition 5.2.2 with Lipschitz constant δ, and let $d(x', x_n, t) = x_n - \Gamma(x', t)$. Then, there exists $\varphi : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \varphi_t - \mathcal{L}\varphi &= 0 \quad \text{in} \quad \Omega \\ \varphi &= 0 \quad \text{on} \quad \partial_\Gamma \Omega, \end{cases}$$

$\varphi \geq 0$, $\|\varphi\|_{L^\infty(Q_1)} = 1$,

$$\frac{1}{24} d^{1+\varepsilon} \leq \varphi \leq 64 d^{1-\varepsilon},$$

and for all $0 < r_1 < r_2 \leq 1$,

$$\frac{\sup_{Q_{r_1}} \varphi}{\sup_{Q_{r_2}} \varphi} \geq \frac{1}{8} \left(\frac{r_1}{r_2}\right)^{1+\varepsilon}.$$

We start by constructing solutions with a controlled growth.

Lemma 5.4.2. *Let $\varepsilon \in (0, 1)$, and let $\mathcal{L}$ be a non-divergence form operator as in (5.1). There exists $\delta_1 \in (0, \varepsilon)$, only depending on the dimension, ε and the ellipticity constants, such that the following holds.*

Let $R = 2^{1/\varepsilon}$, let Ω be a parabolic Lipschitz domain in Q_R in the sense of Definition 5.2.2 with Lipschitz constant δ_1, and let $d = x_n - \Gamma(x', t)$. Then, there exists $\varphi : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \varphi_t - \mathcal{L}\varphi &= 0 \quad \text{in} \quad \Omega \\ \varphi &= 0 \quad \text{on} \quad \partial_\Gamma \Omega, \end{cases}$$

$\varphi \geq 0$, $\|\varphi\|_{L^\infty(Q_1)} = 1$, *and*

$$\frac{1}{24} d^{1+\varepsilon} \leq \varphi \leq 64 d^{1-\varepsilon} \quad \text{in } Q_R.$$

In particular, $\|\varphi\|_{L^\infty(Q_r)} \leq 128 r^{1-\varepsilon}$ *for all* $r \in [1, R]$.

Proof. First, by the same computations in Lemmas 5.3.4 and 5.3.5,

$$\begin{aligned} (\partial_t - \mathcal{L}) d^{1-\varepsilon} &\geq (1-\varepsilon) d^{-1-\varepsilon}(-C\delta_1 + C'\varepsilon) \geq 0 \\ (\partial_t - \mathcal{L}) d^{1+\varepsilon} &\leq (1+\varepsilon) d^{-1+\varepsilon}(C\delta_1 - C'\varepsilon) \leq 0, \end{aligned}$$

provided that δ_1 is small enough. Assume without loss of generality that $\delta_1 \in (0, 1/6)$. Then, since

$$d(x, t) \leq \frac{3}{2}(x_n - \Gamma(x', t)) \leq \frac{3}{2}(R + 2\delta_1 R) \leq 2R,$$

it follows that

$$d^{1+\varepsilon} \leq (2R)^\varepsilon d \leq (2R)^{2\varepsilon} d^{1-\varepsilon}.$$

Now, let $\tilde{\varphi}$ be the solution to

$$\begin{cases} \tilde{\varphi}_t - \mathcal{L}\tilde{\varphi} &= 0 \quad &\text{in} \quad \Omega \\ \tilde{\varphi} &= (2R)^\varepsilon d \quad &\text{on} \quad \partial_p \Omega. \end{cases}$$

By the comparison principle, it follows that

$$d^{1+\varepsilon} \leq \tilde{\varphi} \leq (2R)^{2\varepsilon} d^{1-\varepsilon}.$$

Then, by Lemma 5.3.2,

$$\|\tilde{\varphi}\|_{L^\infty(Q_1)} \le (2R)^{2\varepsilon}\|d^{1-\varepsilon}\|_{L^\infty(Q_1)} \le 4R^{2\varepsilon}\frac{3}{2} = 24,$$
$$\|\tilde{\varphi}\|_{L^\infty(Q_1)} \ge \|d^{1+\varepsilon}\|_{L^\infty(Q_1)} \ge \frac{1}{4}.$$

Let now

$$\varphi := \frac{\tilde{\varphi}}{\|\tilde{\varphi}\|_{L^\infty(Q_1)}}.$$

The first conclusion follows from the previous estimate. For the second one, notice that for $r \ge 1$,

$$\|\varphi\|_{L^\infty(Q_r)} \le 64\|d^{1-\varepsilon}\|_{L^\infty(Q_r)} \le 64\left(\frac{3}{2}r(1+2\delta_1)\right)^{1-\varepsilon} \le 128r^{1-\varepsilon}.$$

□

This special solutions satisfy the following estimate.

Lemma 5.4.3. *Let $\varepsilon \in (0, \alpha_0)$ and let $\mathcal{L}$ be a non-divergence form operator as in (5.1). There exists an integer $n_0 > 1/\varepsilon$, only depending on ε, the dimension and the ellipticity constants, such that the following holds.*

Let $R_0 = 2^{n_0}$, let Ω be a parabolic Lipschitz domain in Q_{R_0} in the sense of Definition 5.2.2 with Lipschitz constant $1/n_0$, and let $d = x_n - \Gamma(x', t)$. Let $\varphi : \Omega \to \mathbb{R}$ satisfy the following properties:

$$\begin{cases} \varphi_t - \mathcal{L}\varphi = 0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial_\Gamma\Omega \\ \varphi \ge 0 & \\ \|\varphi\|_{L^\infty(Q_{2^k})} \le 128\cdot 2^{k(1+1/n_0+\varepsilon)} & \forall\ k \in \{0,\dots,n_0\} \\ \|\varphi\|_{L^\infty(Q_1)} = 1 & \end{cases}$$

Then,

$$\sup_{Q_{1/2}} \varphi \ge \left(\frac{1}{2}\right)^{1+1/n_0+\varepsilon}.$$

Proof. Let us proceed by contradiction: assume there does not exist n_0 satisfying the conclusion. Then, by Lemma 5.4.2 with $\varepsilon = 1/n_0$, there exist $n_k \uparrow \infty$, $\mathcal{L}_k$ non-divergence form operators, Ω_k parabolic Lipschitz domains in Q_{R_k} (with Lipschitz constant $1/n_k$ and $R_k = 2^{n_k}$), and $\varphi_k : \Omega_k \to \mathbb{R}$ such that

$$\begin{cases} (\partial_t - \mathcal{L}_k)\varphi_k = 0 & \text{in } \Omega_k \\ \varphi_k = 0 & \text{on } \partial_\Gamma\Omega_k \\ \varphi_k \ge 0 & \\ \|\varphi_k\|_{L^\infty(Q_{2^k})} \le 128\cdot 2^{k(1+1/n_k+\varepsilon)} & \forall\ k \in \{0,\dots,n_k\} \\ \|\varphi_k\|_{L^\infty(Q_1)} = 1 & \end{cases}$$

while also satisfying

$$\sup_{Q_{1/2}} \varphi_k < \left(\frac{1}{2}\right)^{1+1/n_k+\varepsilon}.$$

Then, by Proposition 5.3.1, for all $r \geq 1$

$$\|\varphi_k\|_{C_p^{0,\alpha}(Q_r)} \leq C(r),$$

for sufficiently large k, and therefore by Arzelà-Ascoli $\varphi_k \to \varphi_0$ locally uniformly, up to a subsequence.

Therefore, by Proposition 5.2.7, φ_0 is a viscosity solution to

$$(\partial_t - \mathcal{M}^+)\varphi_0 \leq 0 \leq (\partial_t - \mathcal{M}^-)\varphi_0 \quad \text{in } \{x_n > 0\}$$

with $\varphi_0 = 0$ on $\{x_n = 0\}$, $\varphi_0 \geq 0$, $\|\varphi_0\|_{L^\infty(Q_1)} = 1$, and the growth control $\|\varphi_0\|_{L^\infty(Q_{2^k})} \leq 128 \cdot 2^{k(1+\varepsilon)}$ for all $k \in \mathbb{N}$.

Hence, by Theorem 5.2.8, $\varphi_0 = (x_n)_+$, contradicting the fact that

$$\frac{1}{2} = \sup_{Q_{1/2}} \varphi_0 \leq \limsup_{k\to\infty} \sup_{Q_{1/2}} \varphi_k \leq \lim_{k\to\infty} \left(\frac{1}{2}\right)^{1+1/n_k+\varepsilon} = \frac{1}{2^{1+\varepsilon}} < \frac{1}{2}.$$

□

The next step is to iterate the inequality to obtain the following.

Lemma 5.4.4. *Under the hypotheses of Lemma 5.4.3, for all* $0 < r_1 < r_2 \leq 1$,

$$\frac{\sup_{Q_{r_1}} \varphi}{\sup_{Q_{r_2}} \varphi} \geq \frac{1}{8}\left(\frac{r_1}{r_2}\right)^{1+1/n_0+\varepsilon}.$$

Proof. Assume without loss of generality that $\varepsilon \in (0, \frac{1}{2})$. Let us first prove by induction that

$$\sup_{Q_{2^{-k}}} \varphi \geq 2^{-k(1+\varepsilon+1/n_0)}.$$

It suffices to prove that the function

$$\bar{\varphi}(x,t) := \frac{\varphi(x/2, t/4)}{\sup_{Q_{1/2}} \varphi}$$

also satisfies the hypotheses of Lemma 5.4.3, and then the argument can be iterated. By construction, $(\partial_t - \bar{\mathcal{L}})\bar{\varphi} = 0$ in Q_{2R_0}, $\bar{\varphi} \geq 0$ and $\|\bar{\varphi}\|_{L^\infty(Q_1)} = 1$. Additionally, by Lemma 5.4.3, for all $k \in \{1, \ldots, n_0 + 1\}$,

$$\sup_{Q_{2^k}} \bar{\varphi} = \frac{\sup_{Q_{2^{k-1}}} \varphi}{\sup_{Q_{1/2}} \varphi} \leq 128 \cdot 2^{(k-1)(1+1/n_0+\varepsilon)} \cdot 2^{1+1/n_0+\varepsilon} = 128 \cdot 2^{k(1+1/n_0+\varepsilon)}.$$

On the other hand, the reasoning with $\bar{\varphi}$ implies that for any $k \in \mathbb{N}$,

$$\tilde{\varphi}(x,t) := \frac{\varphi(x/2^k, t/4^k)}{\sup_{Q_{2^{-k}}} \varphi}$$

also satisfies the hypotheses of Lemma 5.4.3, and hence, by the first part of the proof

$$\frac{\sup_{2^{-k-m}} \varphi}{\sup_{2^{-k}} \varphi} = \sup_{Q_{2^{-m}}} \tilde{\varphi} \geq 2^{-m(1+\varepsilon+1/n_0)}.$$

Now, choose k and m in such a way that $2^{-k-m} \leq r_1 < 2^{-k-m+1}$ and $2^{-k-1} < r_2 \leq 2^{-k}$. Then,

$$\frac{\sup_{Q_{r_1}} \varphi}{\sup_{Q_{r_2}} \varphi} \geq \frac{\sup_{2^{-k-m}} \varphi}{\sup_{2^{-k}} \varphi} \geq 2^{-m(1+\varepsilon+1/n_0)} > \left(\frac{r_1}{4r_2}\right)^{1+\varepsilon+1/n_0} > \frac{1}{8}\left(\frac{r_1}{r_2}\right)^{1+\varepsilon+1/n_0}.$$

□

Finally we can combine Lemma 5.4.2 with Lemma 5.4.4 to prove our target result.

Proof of Proposition 5.4.1. Choose n_0 from Lemma 5.4.3 with $\varepsilon/2$ instead of ε. Then, the function introduced in Lemma 5.4.2 with $\varepsilon = 1/n_0$ satisfies the hypotheses of Lemma 5.4.3, and the conclusion follows by Lemma 5.4.4 (with $\varepsilon/2$ instead of ε). □

5.5 Proof of the boundary Harnack

The main ingredient in the proof of the boundary Harnack is the following expansion result.

Proposition 5.5.1. *Let $\alpha \in (0, \alpha_0)$, and let $\mathcal{L}$ be a non-divergence form operator as in (5.1). There exists $\varepsilon_0 \in (0,1)$, only depending on α, the dimension and the ellipticity constants, such that the following holds.*

Let Ω be a parabolic Lipschitz domain in Q_1 in the sense of Definition 5.2.2 with Lipschitz constant ε_0. Let $d(x', x_n, t) = x_n - \Gamma(x', t)$, and let u be a solution to

$$\begin{cases} u_t - \mathcal{L}u &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial_\Gamma \Omega, \end{cases}$$

and assume that $\|u\|_{L^\infty(Q_1)} \leq 1$, and that $f = g + h$ with

$$\|d^{1-\alpha} g\|_{L^\infty(Q_1)} + \|d^{-1/(n+1)-\alpha} h\|_{L^{n+1}(Q_1)} \leq 1.$$

Then, for each $r \in (0,1]$ there exists $K_r \in \mathbb{R}$ such that $|K_r| \leq C$ and

$$\|u - K_r \varphi\|_{L^\infty(Q_r)} \leq C r^{1+\alpha},$$

where φ is the near-linear solution introduced in Proposition 5.4.1 and C depends only on α, the dimension and the ellipticity constants.

Before proving the expansion, we need to introduce the following growth estimate for blow-ups (cf. [19, Lemma 4.4]), which is independent of the PDE and valid for general functions.

Lemma 5.5.2. *Let $\beta > \gamma > 0$. For every $j \in \mathbb{N}$, let $\Omega_j \subset \mathbb{R}^{n+1}$, and let $u_j, \varphi_j : \Omega_j \to \mathbb{R}$ such that $\|u_j\|_{L^\infty(Q_1)} \le 1$, $\|\varphi_j\|_{L^\infty(Q_1)} = 1$, and, for every $0 < r_1 < r_2 < 1$,*

$$\frac{\sup_{Q_{r_1}} \varphi}{\sup_{Q_{r_2}} \varphi} \ge c_1 \left(\frac{r_1}{r_2}\right)^\gamma.$$

Let $K_{r,j} \in \mathbb{R}$ for every $r \in (0,1]$ and $j \in \mathbb{N}$, and assume that

$$\sup_{j\in\mathbb{N}} |K_{r,j}| < \infty$$

and

$$\sup_{r\in(0,1]} \theta(r) = \infty,$$

where

$$\theta(r) := \sup_{\rho\in(r,1]} \sup_{j\in\mathbb{N}} \rho^{-\beta} \|u_j - K_{\rho,j}\varphi_j\|_{L^\infty(Q_\rho)}.$$

Then, there exist sequences $\rho_m \downarrow 0$ and j_m such that

$$\rho_m^{-\beta} \|u_{j_m} - K_{\rho_m,j_m}\varphi_{j_m}\|_{L^\infty(Q_{\rho_m})} \ge \frac{1}{2}\theta(\rho_m),$$

and

$$w_m := \frac{u_{j_m}(\rho_m x, \rho_m^2 t) - K_{\rho_m,j_m}\varphi_{j_m}(\rho_m x, \rho_m^2 t)}{\|u_{j_m}(\rho_m x, \rho_m^2 t) - K_{\rho_m,j_m}\varphi_{j_m}(\rho_m x, \rho_m^2 t)\|_{L^\infty(Q_1)}}$$

satisfies

$$\|w_m\|_{L^\infty(Q_R)} \le CR^\beta \;\forall R \in [1, 1/\rho_m).$$

Moreover, for every $0 < r_1 < r_2 < 1$,

$$\|(K_{r_2,j} - K_{r_1,j})\varphi_j\|_{L^\infty(Q_{r_2})} \le Cr_2^\beta \theta(r_1).$$

The constant C depends only on β, γ, and c_1.

We defer the proof to Appendix 5.10. Using Lemma 5.5.2, we can prove the expansion:

Proof of Proposition 5.5.1. We divide the proof into four steps.

Step 1. We reason by contradiction and construct a blow-up sequence. Let us prove first the following modified claim:

Claim. For every $r \in (0,1]$, there exists K_r with $|K_r| \le C_0 r^{-\alpha}$ such that

$$\|u - K_r\varphi\|_{L^\infty(Q_r)} \le Cr^{1+\alpha},$$

with C_0 to be chosen later.

If we assume the claim does not hold, there are sequences u_j, φ_j, and Ω_j (with parabolic Lipschitz constant less than $1/j$), such that

$$\begin{cases} (\partial_t - \mathcal{L}_j)u_j = f_j & \text{in } \Omega_j \\ u_j = 0 & \text{on } \partial_\Gamma \Omega_j, \end{cases}$$

with $f_j = g_j + h_j$ such that

$$\|d^{1-\alpha} g_j\|_{L^\infty(Q_1)} + \|d^{-1/(n+1)-\alpha} h_j\|_{L^{n+1}(Q_1)} \le 1,$$

and

$$\|u_j - K_{r,j}\varphi_j\|_{L^\infty(Q_{r_j})} \ge j r_j^{1+\alpha},$$

where we choose

$$K_{r,j} := \frac{\int_{Q_r} u_j \varphi_j}{\int_{Q_r} \varphi_j^2}.$$

Then, by Proposition 5.4.1 with $\varepsilon = \alpha/2$,

$$\frac{\sup_{Q_{r_1}} \varphi_j}{\sup_{Q_{r_2}} \varphi_j} \ge \frac{1}{8}(r_1/r_2)^{1+\alpha/2},$$

and by Propositions 5.3.9 and 5.4.1,

$$|K_{r,j}| \le \frac{\left(\int_{Q_r} u_j^2\right)^{1/2}}{\left(\int_{Q_r} \varphi_j^2\right)^{1/2}} \le \frac{C r^{1-\alpha/2}}{c r^{1+\alpha/2}} =: C_0 r^{-\alpha},$$

where we choose the constant C_0 from this computation.

Then, by Lemma 5.5.2 with $\gamma = 1 + \alpha/2$ and $\beta = 1 + \alpha$, there exists a sequence $\rho_m \downarrow 0$ such that

$$w_m := \frac{u_{j_m}(\rho_m x, \rho_m^2 t) - K_{\rho_m, j_m}\varphi_{j_m}(\rho_m x, \rho_m^2 t)}{\|u_{j_m}(\rho_m x, \rho_m^2 t) - K_{\rho_m, j_m}\varphi_{j_m}(\rho_m x, \rho_m^2 t)\|_{L^\infty(Q_1)}}$$

satisfies $\|w_m\|_{L^\infty(Q_1)} = 1$,

$$\|w_m\|_{L^\infty(Q_R)} \le C R^{1+\alpha}, \ \forall R \in [1, 1/\rho_m),$$

and

$$\int_{Q_1} w_m(x,t)\varphi_{j_m}(\rho_m x, \rho_m^2 t) = 0$$

from the choice of $K_{\rho_m, j}$.

Step 2. We will prove that $w_m \to (x_n)_+$ locally uniformly along a subsequence.

First, by the construction of w_m, we have (omitting the dependence of f on j_m)

$$(\partial_t - \mathcal{M}^+) w_m \le (\partial_t - \mathcal{L}_m) w_m \le \frac{2\rho_m^{1-\alpha}}{\theta(\rho_m)} |f(\rho_m x, \rho_m^2 t)| \quad \text{in } \tilde{\Omega}_{j_m},$$

where $\mathcal{L}_m$ is the corresponding scaled operator, that has the same ellipticity constants, and

$$\tilde{\Omega}_{j_m} := \{(x,t) : (\rho_m x, \rho_m^2 t) \in \Omega_{j_m}\}.$$

Note that $\tilde{\Omega}_{j_m}$ has Lipschitz constant lower or equal to $1/j_m$.

Let d be the regularized distance in the domain Ω_{j_m}, $\tilde{d}$ the regularized distance in $\tilde{\Omega}_{j_m}$, and let us omit the dependence of g, h on j. Then, using Lemma 5.3.3,

$$\begin{aligned}\left\|\tilde{d}^{1-\alpha}\frac{2\rho_m^{1-\alpha}}{\theta(\rho_m)}|g(\rho_m x, \rho_m^2 t)|\right\|_{L^\infty(Q_{1/\rho_m})} &\le \left\|\frac{2\rho_m^{1-\alpha}}{\theta(\rho_m)}\left[\left(\frac{3d}{\rho_m}\right)^{1-\alpha}|g|\right](\rho_m x, \rho_m^2 t)\right\|_{L^\infty(Q_{1/\rho_m})} \\ &\le \frac{6}{\theta(\rho_m)}\|d^{1-\alpha}|g|\|_{L^\infty(Q_1)}.\end{aligned}$$

Similarly, by the scaling of the L^{n+1} norm,

$$\begin{aligned}&\left\|\tilde{d}^{-1/(n+1)-\alpha}\frac{2\rho_m^{1-\alpha}}{\theta(\rho_m)}|h(\rho_m x, \rho_m^2 t)|\right\|_{L^{n+1}(Q_{1/\rho_m})} \\ &\le \left\|\frac{2\rho_m^{1-\alpha}}{\theta(\rho_m)}\left[\left(\frac{d}{3\rho_m}\right)^{-1/(n+1)-\alpha}|h|\right](\rho_m x, \rho_m^2 t)\right\|_{L^{n+1}(Q_{1/\rho_m})} \\ &\le \frac{18\rho_m^{(n+2)/(n+1)}}{\theta(\rho_m)}\|(d^{-1/(n+1)-\alpha}|h|)(\rho_m x, \rho_m^2 t)\|_{L^{n+1}(Q_{1/\rho_m})} \\ &= \frac{18}{\theta(\rho_m)}\|d^{-1/(n+1)-\alpha}|h|\|_{L^{n+1}(Q_1)}.\end{aligned}$$

Therefore,

$$(\partial_t - \mathcal{M}^+)w_m \le |g_m| + |h_m| \quad \text{in} \quad Q_{1/\rho_m} \cap \tilde{\Omega}_{j_m},$$

with

$$\|\tilde{d}^{1-\alpha}g_m\|_{L^\infty(Q_{1/\rho_m})} + \|\tilde{d}^{-1/(n+1)-\alpha}h_m\|_{L^{n+1}(Q_{1/\rho_m})} \le \frac{18}{\theta(\rho_m)}.$$

Analogously,

$$(\partial_t - \mathcal{M}^-)w_m \ge -|g_m| - |h_m| \quad \text{in} \quad Q_{1/\rho_m} \cap \tilde{\Omega}_{j_m}.$$

Moreover, $w_m = 0$ on $\partial_\Gamma \tilde{\Omega}_{j_m}$, and, for every $R \ge 1$, $\|w_m\|_{L^\infty(Q_R)} \le CR^{1+\alpha}$ for sufficiently large m. Hence, by Proposition 5.3.1,

$$\|w_m\|_{C_p^{0,\alpha}(Q_R)} \le C(R),$$

uniformly in m, for m large enough. Then, by Arzelà-Ascoli and Proposition 5.2.7, we obtain that

$$w_m \to w \in C(\mathbb{R}^{n+1}),$$

locally uniformly along a subsequence, where w is a viscosity solution of

$$\begin{cases} w_t - \mathcal{M}^+ w \;\le\; 0 \;\le\; w_t - \mathcal{M}^- w & \text{in} \quad \{x_n > 0\} \\ \qquad\qquad w \;=\; 0 & \text{on} \quad \{x_n = 0\}, \end{cases}$$

$\|w\|_{L^\infty(Q_1)} = 1$ and $\|w\|_{L^\infty(Q_R)} \le CR^{1+\alpha}$ for all $R \ge 1$. Therefore, by Theorem 5.2.8, $w = (x_n)_+$.

Step 3. Let us consider the functions

$$\tilde{\varphi}_{j_m}(x,t) := \frac{\varphi_{j_m}(\rho_m x, \rho_m^2 t)}{\|\varphi_{j_m}\|_{L^\infty(Q_{\rho_m})}}.$$

Then, $\|\tilde{\varphi}_{j_m}\|_{L^\infty(Q_1)} = 1$, and, by Proposition 5.4.1 with $\varepsilon = \alpha/2$, for all $1 \le R \le 1/\rho_m$,

$$\|\tilde{\varphi}_{j_m}\|_{L^\infty(Q_R)} \le 8R^{1+\alpha/2}.$$

Finally, by the same arguments as in Step 2, $\tilde{\varphi}_{j_m} \to (x_n)_+$ locally uniformly along a subsequence.

Step 4. We have $w_m \to (x_n)_+$ and $\tilde{\varphi}_m \to (x_n)_+$ locally uniformly. Now, recall that by the choice of $K_{r,j}$ in the construction of w_m,

$$\int_{Q_1} w_m \tilde{\varphi}_m = 0,$$

and passing to the limit,

$$\int_{Q_1} (x_n)_+^2 = 0,$$

which is a contradiction. Therefore, for every $r \in (0,1]$, there exists $|K_r| \le C_0 r^{-\alpha}$ such that

$$\|u - K_r\varphi\|_{L^\infty(Q_r)} \le Cr^{1+\alpha}.$$

This is enough for $r \in (\frac{1}{2}, 1]$. For smaller values of r, observe that

$$\begin{aligned} |K_r - K_{r/2}|(r/4)^{1+\alpha/2} &\le \|(K_r - K_{r/2})\varphi\|_{L^\infty(Q_{r/2})} \\ &\le \|u - K_r\varphi\|_{L^\infty(Q_r)} + \|u - K_{r/2}\varphi\|_{L^\infty(Q_{r/2})} \le Cr^{1+\alpha}. \end{aligned}$$

It follows that $|K_r - K_{r/2}| \le Cr^{\alpha/2}$. Then, for $r \le \frac{1}{2}$ we can write $r = 2^{-a}r_0$, with $r_0 \in (\frac{1}{2}, 1]$, and estimate

$$|K_r| \le |K_{r_0}| + \sum_{i=0}^{a-1} |K_{2^{-i}r_0} - K_{2^{-i-1}r_0}| \le C_0 r_0^{-\alpha} + C\sum_{i=0}^{a-1} (2^{-i}r_0)^{\alpha/2} \le C.$$

□

Finally, we prove our main result.

Proof of Theorem 5.1.2. First, we will use a similar strategy to the proof of Proposition 5.3.1 to estimate the Hölder seminorm of the quotient. Let $\varepsilon > 0$ in Proposition 5.4.1 such that $\gamma = \alpha - 7\varepsilon$. Recall that α is chosen in Remark 5.1.3.

Let $p = (y', y_n, s)$ and $\rho \in (0, \frac{1}{16})$ such that $Q_{2\rho}(p) \subset \Omega \cap Q_{5/8}$, and let

$$R := \max\left\{\rho, \frac{y_n - \Gamma(y', s)}{3}\right\}.$$

Then, we distinguish two cases (cf. Proposition 5.3.1).

Case 1. $R \ge \frac{1}{16}$. Then, $Q_{1/8}(p) \subset \Omega \cap Q_{3/4}$, and for all $(x', x_n, t) \in Q_{1/8}(p)$, $x_n \ge \Gamma(x', t) + \frac{1}{16}$, provided that the Lipschitz constant of the domain is small enough. By Proposition 5.3.9, $v \ge cm > 0$ in $Q_{1/8}(p)$. Furthermore, by Theorem 5.2.5, $\|u\|_{C_p^{0,\gamma}(Q_{1/8}(p))} \le C$ and $\|v\|_{C_p^{0,\gamma}(Q_{1/8}(p))} \le C$. Therefore,

$$\begin{aligned} \left\|\frac{u}{v}\right\|_{C_p^{0,\gamma}(Q_\rho(p))} &\le \left\|\frac{u}{v}\right\|_{C_p^{0,\gamma}(Q_{1/8}(p))} \\ &\le \frac{\|u\|_{C_p^{0,\gamma}(Q_{1/8}(p))}\|v\|_{L^\infty(Q_{1/8}(p))} + \|u\|_{L^\infty(Q_{1/8}(p))}\|v\|_{C_p^{0,\gamma}(Q_{1/8}(p))}}{\inf_{Q_{1/8}(p)} v^2} \le Cm^{-2}. \end{aligned}$$

Case 2. $R < \frac{1}{16}$. Then, for all $(x', x_n, t) \in Q_{3R/2}(p)$,

$$\Gamma(x', t) + \frac{R}{2} \leq x_n \leq \Gamma(x', t) + 5R.$$

Let φ be the special solution defined in Proposition 5.4.1, centered at $(y', \Gamma(y', s), s)$. Then, by a translation of Proposition 5.5.1, $w_1 = u - K_u\varphi$ and $w_2 = v - K_v\varphi$ satisfy

$$\|w_1\|_{L^\infty(Q_R(p))} \leq CR^{1+\alpha} \quad \text{and} \quad \|w_2\|_{L^\infty(Q_R(p))} \leq CR^{1+\alpha}.$$

Using that d is comparable to R in $Q_R(p)$, the right-hand side of the equation for u can be estimated as

$$\begin{aligned} \|f_1\|_{L^{n+1}(Q_R(p))} &\leq \|g_1\|_{L^{n+1}(Q_R(p))} + \|h_1\|_{L^{n+1}(Q_R(p))} \\ &\lesssim R^{(n+2)/(n+1)} R^{\alpha-1} \|d^{1-\alpha} g_1\|_{L^\infty(Q_R(p))} \\ &\quad + R^{\alpha+1/(n+1)} \|d^{-1/(n+1)-\alpha} h_1\|_{L^{n+1}(Q_R(p)))} \leq CR^{\alpha+1/(n+1)}, \end{aligned}$$

and analogously, in the equation for v, $\|f_2\|_{L^{n+1}(Q_R(p))} \leq CmR^{\alpha+1/(n+1)}$. Thus, by the interior estimates in Theorem 5.2.5, and the growth of v and φ, (see Propositions 5.3.9 and 5.4.1),

$$\begin{aligned} [w_1]_{C_p^{0,\gamma}(Q_R(p))} &\leq CR^{-\gamma}(CR^{1+\alpha} + CR^{1+\alpha}) \leq CR^{1+7\varepsilon} \\ [w_2]_{C_p^{0,\gamma}(Q_R(p))} &\leq CR^{-\gamma}(CR^{1+\alpha} + CmR^{1+\alpha}) \leq CR^{1+7\varepsilon} \\ [v]_{C_p^{0,\gamma}(Q_R(p))} &\leq CR^{-\gamma}(CR^{1-\varepsilon} + CmR^{1+\alpha}) \leq CR^{1-\gamma-\varepsilon} \\ [\varphi]_{C_p^{0,\gamma}(Q_R(p))} &\leq CR^{-\gamma}(CR^{1-\varepsilon}) \leq CR^{1-\gamma-\varepsilon}. \end{aligned}$$

Now, using that $u = w_1 + K_u\varphi$, we estimate first

$$\begin{aligned} [w_1/v]_{C_p^{0,\gamma}(Q_R(p))} &\leq \frac{[w_1]_{C_p^{0,\gamma}(Q_R(p))} \|v\|_{L^\infty(Q_R(p))} + \|w_1\|_{L^\infty(Q_R(p))} [v]_{C_p^{0,\gamma}(Q_R(p))}}{\inf\limits_{Q_R(p)} v^2} \\ &\leq C \frac{R^{1+7\varepsilon} R^{1-\varepsilon} + R^{1+\alpha} R^{1-\gamma-\varepsilon}}{m^2 R^{2(1+\varepsilon)}} \leq Cm^{-2}, \end{aligned}$$

where we used Proposition 5.3.9 again to deduce that $v \geq cmR^{1+\varepsilon}$ in $Q_R(p)$. Then, we estimate

$$\begin{aligned} [\varphi/v]_{C_p^{0,\gamma}(Q_R(p))} &\leq \frac{[v/\varphi]_{C_p^{0,\gamma}(Q_R(p))}}{\inf\limits_{Q_R(p)} (v/\varphi)^2} \leq \frac{[w_2/\varphi]_{C_p^{0,\gamma}(Q_R(p))}}{\inf\limits_{Q_R(p)} (v/\varphi)^2} \\ &\leq \frac{[\varphi]_{C_p^{0,\gamma}} \|w_2\|_{L^\infty} + [w_2]_{C_p^{0,\gamma}} \|\varphi\|_{L^\infty}}{\inf (v/\varphi)^2 \inf \varphi^2} \\ &\leq C \frac{R^{1-\varepsilon-\gamma} R^{1+\alpha} + R^{1+7\varepsilon} R^{1-\varepsilon}}{(mR^{2\varepsilon})^2 (R^{1+\varepsilon})^2} = 2Cm^{-2}, \end{aligned}$$

where we omitted the domain in the second line to improve readability. Therefore

$$[u/v]_{C_p^{0,\gamma}(Q_R(p))} \leq [w_1/v]_{C_p^{0,\gamma}(Q_R(p))} + |K_u| [\varphi/v]_{C_p^{0,\gamma}(Q_R(p))} \leq Cm^{-2}.$$

Combining the three cases, by Lemma 5.2.6, $[u/\varphi]_{C_p^{0,\gamma}(\Omega \cap Q_{1/2})} \leq Cm^{-2}$.

To obtain a bound for $\|u/\varphi\|_{C_p^{0,\gamma}(\Omega\cap Q_{1/2})}$, observe that

$$\left\|\frac{u}{\varphi}\right\|_{L^\infty(Q_{1/2})} \le \frac{u(e_n/4,0)}{v(e_n/4,0)} + [u/\varphi]_{C_p^{0,\gamma}(\Omega\cap Q_{1/2})}\left(\left|x-\frac{1}{4}e_n\right| + |t|^{1/2}\right) \le \frac{1}{cm} + Cm^{-2},$$

where we used that $\|u\|_{L^\infty(Q_1)} \le 1$ and $v(e_n/4,0) \ge cm > 0$ by the interior Harnack. Therefore, $\|u/\varphi\|_{C_p^{0,\gamma}(\Omega\cap Q_{1/2})} \le Cm^{-2}$, as we wanted to prove. □

5.6 Slit domains

Slit domains appear naturally when studying thin obstacle problems. In the case of parabolic slit domains, they appear in the time-dependent Signorini problem, and the boundary Harnack is known to hold in the homogeneous case; see [161, 75].

First, we will introduce a slightly different notation for this section. Given $x \in \mathbb{R}^{n+1}$, we will denote $x' = (x_1, \ldots, x_{n-1})$, i.e. $x = (x', x_n, x_{n+1})$. $B_r(x', x_n)$ will denote the n-dimensional ball of radius r centered at (x', x_n), and $B'_r(x')$ will be the one of $\mathbb{R}^{n-1}$. We also introduce the *slit parabolic cylinders*:

$$Q_r(x,t) := B'_r(x') \times (x_n - r, x_n + r) \times (x_{n+1} - r, x_{n+1} + r) \times (t - r^2, t) \subset \mathbb{R}^{n+2}.$$

In this section, the domains that we will work with will be the following.

Definition 5.6.1. We say Ω is a parabolic Lipschitz slit domain in Q_R with Lipschitz constant L if $\Omega = Q_R \setminus E$, where

$$E = \Big\{(x', x_n, 0, t) \in Q_R \mid x_n \le \Gamma(x', t)\Big\},$$

and $\Gamma : B'_R \times [-R^2, 0] \to \mathbb{R}$, with $\Gamma(0,0) = 0$ and $\|\Gamma\|_{C_p^{0,1}} \le L$.

We will say that E is the lateral boundary of Ω, and write $\partial_\Gamma\Omega := E$. The parabolic boundary will be defined as

$$\partial_p\Omega := \partial_\Gamma\Omega \cup \Big(\overline{\Omega} \cap \partial Q_R \cap \{t < 0\}\Big).$$

The goal of this section is to prove Theorem 5.1.6.

5.6.1 Growth and boundary regularity

In this section, we will follow the same scheme that in Section 5.3 to obtain growth estimates and regularity up to the boundary of solutions to the heat equation in slit Lipschitz domains. The final goal of the section is to prove the following:

Proposition 5.6.2. *Let $\gamma \in (0, \frac{1}{2})$. There exists $L_0 > 0$, depending only on γ and the dimension, such that the following holds.*

Let Ω be a parabolic slit domain in Q_1 as in Definition 5.6.1 with Lipschitz constant $L \le L_0$. Let u be a solution to

$$\begin{cases} u_t - \Delta u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial_\Gamma\Omega. \end{cases}$$

Assume that $\|f\|_{L^{n+2}(Q_1)} \le K_0$. *Then,*

$$\|u\|_{C_p^{0,\gamma}(Q_{1/2})} \le C(\|u\|_{L^\infty(Q_1)} + K_0).$$

The constant C *depends only on* γ *and the dimension.*

We begin introducing parabolic homogeneous solutions in *parabolic slit cones*, which will play the role of the powers of the regularized distance in Section 5.3.

Proposition 5.6.3 (cf. [103, Lemma 5.8]). *There exists* $\varepsilon_0 > 0$, *only depending on the dimension, such that for all* $\varepsilon \in (0, \varepsilon_0)$, *there exist sufficiently small* $\eta_-, \eta_+ > 0$, *only depending on the dimension and* ε, *such that there exist unique positive solutions of*

$$\partial_t \varphi_- - \Delta \varphi_- = 0 \text{ in } Q_1 \setminus \mathcal{C}^-_{\eta_-} \quad \text{and} \quad \partial_t \varphi_+ - \Delta \varphi_+ = 0 \text{ in } Q_1 \setminus \mathcal{C}^+_{\eta_+},$$

parabolically homogeneous of degree $\frac{1}{2} \pm \varepsilon$, *i.e.*

$$\varphi_-(\lambda x, \lambda^2 t) = \lambda^{\frac{1}{2}-\varepsilon} \varphi_-(x,t) \quad \text{and} \quad \varphi_+(\lambda x, \lambda^2 t) = \lambda^{\frac{1}{2}+\varepsilon} \varphi_+(x,t) \quad \forall \lambda > 0,$$

such that $\|\varphi_-\|_{L^\infty(Q_1)} = \|\varphi_+\|_{L^\infty(Q_1)} = 1$, *where*

$$C^+_{\eta_+} := \{x_n \le \eta_+(|x'| + |t|^{1/2}), x_{n+1} = 0\}$$

and

$$C^-_{\eta_-} := \{x_n \le -\eta_-(|x'| + |t|^{1/2}), x_{n+1} = 0\}.$$

Moreover, $\eta_- \to 0$ *and* $\eta_+ \to 0$ *monotonically as* $\varepsilon \to 0$, *and there exists* $m > 0$ *such that*

$$\varphi_\pm \ge m \text{ in } Q_2 \cap \left\{ |x_{n+1}| \ge \frac{1}{n+1} \right\}.$$

We defer the proof to Appendix 5.10. Notice that φ_- and φ_+ satisfy the following Hopf-type estimate.

Lemma 5.6.4. *Let* $\varphi_\pm$ *be as in Proposition 5.6.3. Then,*

$$\varphi_\pm \ge c|x_{n+1}| \quad \text{in } Q_{1/2},$$

for a dimensional constant $c > 0$.

Proof. It follows from Proposition 5.6.3 and Hopf's lemma. □

Now, we proceed with the same strategy as in Lemmas 5.3.6 an 5.3.8 to obtain the desired bounds. We start with the upper bound:

Lemma 5.6.5. *Let* $\varepsilon \in (0, 1/2)$. *There exist sufficiently small* $\eta > 0$ *and sufficiently large* $K > 0$, *only depending on the dimension and* ε, *such that the following holds.*

Let u *be a solution to*

$$\begin{cases} u_t - \Delta u & \le & f & \text{in} & \Omega \\ u & \le & 1 & \text{on} & \partial_p \Omega \\ u & \le & 0 & \text{on} & \partial_\Gamma \Omega \end{cases}$$

where $\|f\|_{L^{n+2}(\Omega)} \le \eta$, *and* Ω *is a parabolic slit domain in* Q_1 *in the sense of Definition 5.6.1 with Lipschitz constant* η. *Then,*

$$\|u\|_{L^\infty(Q_r)} \le K r^{1/2-\varepsilon}, \quad \forall r \in (0, 1).$$

Proof. We will use the comparison principle with a barrier. Assume first that $f = 0$.

Let φ_- be as in Proposition 5.6.3 with $\varepsilon/2$ instead of ε, let $\eta = \eta_-$ and define

$$v := a\varphi_- + |x'|^2 + x_n^2 - 2nx_{n+1}^2 - 2nt.$$

First, by Lemma 5.6.4 and letting $a = (2n+1)/c$,

$$a\varphi_- \geq ac|x_{n+1}| \geq (2n+1)|x_{n+1}| \quad \text{in } Q_1.$$

Then, v satisfies $v_t - \Delta v = 0$ in $Q_1 \setminus \mathcal{C}_{\eta_-}^- \supset \Omega$. Moreover, on the parabolic boundary of Ω we can distinguish the following cases:

- When $t = -1$, $v \geq a\varphi_- + 2n - 2nx_{n+1}^2$.
- When $|x'| = 1$, and when $|x_n| = 1$, $v \geq a\varphi_- + 1 - 2nx_{n+1}^2$.
- When $|x_{n+1}| = 1$, $v \geq a\varphi_- - 2n$.
- On $\partial_\Gamma\Omega \subset \{x_{n+1} = 0\}$, $v \geq 0$.

To treat the first two cases, note that

$$a\varphi_- + 1 - 2nx_{n+1}^2 \geq 1 + (2n+1)|x_{n+1}| - 2nx_{n+1}^2 \geq 1.$$

Then, when $|x_{n+1}| = 1$,

$$a\varphi_- - 2n \geq 1.$$

Therefore, by the comparison principle, $u \leq v$ in Ω. To include the right-hand side, notice that we can always write $u = u_0 + u_f$, where

$$\begin{cases} (\partial_t - \Delta)u_0 &= 0 \quad \text{in} \quad \Omega \\ u_0 &= u \quad \text{on} \quad \partial_p\Omega \end{cases} \quad \text{and} \quad \begin{cases} (\partial_t - \Delta)u_f &= f \quad \text{in} \quad \Omega \\ u_f &= 0 \quad \text{on} \quad \partial_p\Omega. \end{cases}$$

Then, by Theorem 5.2.4, and applying the reasoning above to u_0,

$$u = u_0 + u_f \leq v + C\|f\|_{L^{n+2}(Q_1)} \leq v + C\eta.$$

Now, we do an iteration scheme as in Lemma 5.3.6. For that, let us define the rescaled functions

$$u_j(x,t) := \frac{u(\rho^j x, \rho^{2j} t)}{\rho^{j(1/2-\varepsilon)}},$$

with $\rho > 0$ to be chosen later. Now,

$$(\partial_t - \Delta)u_j = \rho^{j(3/2+\varepsilon)} f_j,$$

with $f_j(x,t) := f(\rho^j x, \rho^{2j} t)$. Let $b_j := \|u_j\|_{L^\infty(Q_1)}$. Writing $u = u_0 + u_f$ again,

$$b_0 \leq \|u_0\|_{L^\infty(Q_1)} + \|u_f\|_{L^\infty(Q_1)} \leq \|v\|_{L^\infty(Q_1)} + C\|f\|_{L^{n+2}(Q_1)} \leq a + 4n + 2 + C\eta.$$

We will show by induction that $b_j \leq b_0$ for all $j \geq 0$. Indeed, by the first part of the proof and induction hypothesis,

$$u_j \leq v + C\|\rho^{j(3/2+\varepsilon)} f_j\|_{L^{n+2}(Q_1)},$$

and hence, using that v is a sum of terms with at least parabolic homogeneity $\frac{1-\varepsilon}{2}$,

$$
\begin{aligned}
b_{j+1}\rho^{1/2-\varepsilon} &\le \|v\|_{L^\infty(Q_\rho)} + C\rho^{j(3/2+\varepsilon)}\|f_j\|_{L^{n+2}(Q_1)} \\
&\le b_0\rho^{(1-\varepsilon)/2} + \rho^{j(3/2+\varepsilon-(n+3)/(n+2))}\|f\|_{L^{n+2}(Q_{\rho^j})} \\
&\le b_0\rho^{(1-\varepsilon)/2} + \|f\|_{L^{n+2}(Q_1)} \le b_0\rho^{(1-\varepsilon)/2} + \eta \le b_0\rho^{1/2-\varepsilon},
\end{aligned}
$$

after choosing sufficiently small η and ρ.

Finally, given $r \in (0,1)$, let j be such that $\rho^{j+1} < r \le \rho^j$. Therefore,

$$
\|u\|_{L^\infty(Q_r)} \le \|u\|_{L^\infty(Q_{\rho^j})} = b_j\rho^{j(1/2-\varepsilon)} < b_0(r/\rho)^{1/2-\varepsilon}.
$$

□

We also deduce a lower bound:

Lemma 5.6.6. *Let $\varepsilon \in (0,1/6)$ and $\mu \in (0,1)$. There exist sufficiently small $\eta, k > 0$, only depending on the dimension, μ and ε, such that the following holds.*

Let $u \ge 0$ be a solution to

$$
\begin{cases} u_t - \Delta u \ \ge \ f & \text{in} \quad \Omega \\ \qquad\quad u \ \ge \ 1 & \text{on} \quad \partial_{\rm up}\Omega \end{cases}
$$

where $\|f\|_{L^{n+2}(\Omega)} \le \eta$, Ω is a parabolic slit domain in Q_1 in the sense of Definition 5.6.1 with Lipschitz constant η, and

$$
\partial_{\rm up}\Omega := \{x_n = 1\} \cap \overline{\Omega}.
$$

Then,

$$
u(re, 0) \ge kr^{1/2+\varepsilon}, \quad \forall r \in \left(0, \frac{1}{4}\right),
$$

for all unit vectors $e = \cos(\theta)e_n + \sin(\theta)e_{n+1}$ with $\cos(\theta) \ge -1 + \mu$.

Proof. The proof is very similar to that of Lemma 5.6.5. Assume first that $f = 0$.

Let

$$
\Omega^{(1)} := \left\{(x,t) \in \Omega : |x'| < \frac{1}{2}, |x_n| < \frac{1}{2}, |x_{n+1}| < \frac{1}{\sqrt{2(4n+13)}}, -\frac{1}{2} < t\right\}
$$

and

$$
\partial_{\rm up}\Omega^{(1)} := \overline{\Omega^{(1)}} \cap \left\{|x_{n+1}| = \frac{1}{\sqrt{2(4n+13)}}\right\}.
$$

By an analogous reasoning to the proof of Lemma 5.3.5, $u \ge c_0$ on $\partial_{\rm up}\Omega^{(1)}$, and $u(re,0) \ge c_0$ for all $r \in (\frac{1}{\sqrt{2(4n+13)}}, \frac{1}{2})$, given that η is small enough. However, here c_0 depends on μ.

Let φ_+ be as in Proposition 5.6.3 with $\varepsilon/2$ instead of ε, let $\eta \le \eta_+$ and define

$$
v := c_0\left(\frac{1}{2}\varphi_+ + 2t + (4n+13)x_{n+1}^2 - 4|x'|^2 - 16\left(x_n + \frac{1}{4}\right)_-^2\right).
$$

Then, v satisfies $v_t - \Delta v \le 0$ in $Q_1 \setminus \mathcal{C}^+_{\eta_+}$. Moreover, on the parabolic boundary of $\Omega^{(1)}$ we can distinguish the following cases (recall that $\|\varphi_+\|_{L^\infty(Q_1)} = 1$ and that $|x_{n+1}| \le 1/\sqrt{2(4n+13)}$).

- When $t = -1/2$, $v \leq c_0(1/2 - 1 + 1/2) = 0$.
- When $|x'| = 1/2$, and when $x_n = -1/2$, $v \leq c_0(1/2 + 1/2 - 1) = 0$.
- When $x_n = 1/2$, $v \leq c_0(1/2 + 1/2) = c_0$.
- When $|x_{n+1}| = 1/\sqrt{2(4n+13)}$, $v \leq c_0(1/2 + 1/2) = c_0$.
- On $\partial_\Gamma \Omega \subset C^+_{\eta_+}$, $v \leq 0$.

Therefore, by the comparison principle, $u \geq v$ in $\overline{\Omega^{(1)}}$. The right-hand side can be included in the same way as in Lemma 5.6.5, giving

$$u \geq v - C\|f\|_{L^{n+2}(Q_1)} \geq v - C\eta \quad \text{in } \overline{\Omega^{(1)}}.$$

Now, we do an iteration scheme as in Lemma 5.3.8. For that, let us define the rescaled functions

$$u_j(x,t) := \frac{u(\rho^j x, \rho^{2j} t)}{\rho^{j(1/2+\varepsilon)}},$$

with $\rho > 0$ to be chosen later. Now,

$$(\partial_t - \Delta) u_j = \rho^{j(3/2-\varepsilon)} f_j,$$

with $f_j(x,t) := f(\rho^j x, \rho^{2j} t)$. Let

$$b_j := \inf_{\partial_{\mathrm{up}}\Omega_j} u_j,$$

where Ω_j is the appropriate scaled domain of u_j in Q_1. By hypothesis, $b_0 \geq 1$. We will show by induction that $b_j \geq 1$ for all $j \geq 0$.

Indeed, by the first part of the proof and induction hypothesis,

$$u_j \geq v - C\|\rho^{j(3/2-\varepsilon)} f_j\|_{L^{n+2}(Q_1)},$$

and hence, using the parabolic homogeneity of φ_+, and using the same scaling arguments as in Lemma 5.6.5

$$\begin{aligned} b_{j+1}\rho^{1/2+\varepsilon} &\geq c_0 \frac{\inf_{\partial_{\mathrm{up}}\Omega_j} \varphi_+}{2} \rho^{(1+\varepsilon)/2} - (4n+14)c_0\rho^2 - C\rho^{j(3/2-\varepsilon)}\|f_j\|_{L^{n+2}(\Omega_j^{(1)})} \\ &\geq c_0 c \rho^{(1+\varepsilon)/2} - C\rho^2 - C\rho^{j(3/2-\varepsilon-(n+3)/(n+2))}\eta \geq \rho^{1/2+\varepsilon}, \end{aligned}$$

for sufficiently small η and ρ.

Finally, by the first part of the proof,

$$u_j \geq v - C\eta \quad \text{in } \overline{\Omega^{(1)}},$$

and then

$$u_j \geq \frac{c_0}{2}\varphi_+ - C\eta \quad \text{in } \{t = 0, x' = 0, x_n \geq -1/4\},$$

which in turn implies $u_j(re, 0) \geq c(\mu) r^{1/2+\varepsilon} - C\eta \geq k r^{1/2+\varepsilon}$ for all $r \in [1/(4\rho), 1/4)$. The conclusion follows undoing the scaling. □

Combining Lemmas 5.6.5 and 5.6.6, we can deduce a growth estimate for all solutions.

Proposition 5.6.7. *Let $\varepsilon \in (0, 1/6)$. There exists sufficiently small $\eta > 0$, only depending on ε and the dimension, such that the following holds.*

Let Ω be a parabolic slit domain in Q_1 in the sense of Definition 5.6.1. Let u be a solution to

$$\begin{cases} u_t - \Delta u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial_\Gamma \Omega. \end{cases}$$

Assume that $\|u\|_{L^\infty(Q_1)} \le 1$, and $\|f\|_{L^{n+2}(Q_1)} \le \eta$.

Then,

$$|u| \le C|(x_n - \Gamma(x', t), x_{n+1})|^{1/2-\varepsilon} \quad \text{in} \quad \Omega \cap Q_{3/4}.$$

Moreover, if u is nonnegative, $m = u\left(\frac{e_n}{2}, -\frac{3}{4}\right) > 0$, $\|f\|_{L^{n+2}(Q_1)} \le \eta m$, then

$$u \ge cm|(x_n - \Gamma(x', t), x_{n+1})|^{1/2+\varepsilon} \quad \text{in} \quad \Omega \cap \mathcal{C} \cap Q_{3/4},$$

where $\mathcal{C}$ is the "cone" defined as

$$\mathcal{C} := \left\{(x, t) \in \mathbb{R}^{n+2} \mid x_n - \Gamma(x', t) + 10|x_{n+1}| \ge 0\right\}.$$

The constants C and c are positive and depend only on the dimension, ε, and the ellipticity constants.

Proof. We will follow the same strategy as in Proposition 5.3.9. The proof of the upper bound is exactly the same, using Lemma 5.6.5 instead of Lemma 5.3.6.

For the second estimate, let $\Omega(x_0, t_0) := \Omega \cap Q_{1/8}(x_0, t_0)$ and notice that

$$\bigcup_{(x_0,t_0)\in\partial_\Gamma\Omega\cap Q_{3/4}} \partial_{\text{up}}\Omega(x_0, t_0) \subset E := \overline{B'_{7/8}} \times [1/8 - 2\eta, 1/8 + 2\eta] \times [-1/8, 1/8] \times [-37/64, 0],$$

where $\partial_{\text{up}}\Omega(x_0, t_0) := \overline{\Omega(x_0, t_0)} \cap \{x_n = x_{0,n} + 1/8\}$, analogously to Lemma 5.6.6. The rest follows as in Proposition 5.3.9, but here we use the interior Harnack to see that $u \ge c_1 m$ in

$$F := Q_{3/4} \setminus \{x_n < 1/32 - 2\eta,\ |x_{n+1}| < 1/32\}$$

instead of E. Here we use $\eta < 1/256$ to ensure that there is a uniform positive distance from F to the boundary of Ω. □

Finally, we can deduce the Hölder regularity up to the boundary.

Proof of Proposition 5.6.2. We will use the same strategy as in Proposition 5.3.1. Let us assume without loss of generality that $\|u\|_{L^\infty(Q_1)} \le 1$ and $K_0 = \eta$ (with η from Proposition 5.6.7). Then, by Proposition 5.6.7,

$$|u| \le C(|x_n - \Gamma(x', t)| + |x_{n+1}|)^\gamma \quad \text{in } \Omega \cap Q_{3/4}.$$

Then, we will use interior estimates in combination with Lemma 5.2.6. Let $p = (y', y_n, y_{n+1}, s)$ and $\rho \in (0, \frac{1}{16})$ such that $Q_{2\rho}(p) \subset \Omega \cap Q_{5/8}$, and let

$$R := \max\left\{\rho, \frac{|y_{n+1}|}{3}, \frac{|y_n - \Gamma(y', s)|}{3}\right\}.$$

We distinguish four cases:

Case 1. $R \geq \frac{1}{16}$, and $y_n > \Gamma(y', s) - 3R$ or $|y_{n+1}| \geq \frac{1}{8}$. Then, $Q_{1/8}(p) \subset \Omega \cap Q_{3/4}$. Then, by Theorem 5.2.5 combined with the fact that $\|u\|_{L^\infty(Q_1)} \leq 1$ and $\|f\|_{L^{n+2}(Q_1)} \leq 1$, we obtain

$$[u]_{C_p^{0,\gamma}(Q_\rho(p))} \leq [u]_{C_p^{0,\gamma}(Q_{1/8}(p))} \leq C.$$

Case 2. $R \geq \frac{1}{16}$, $y_n = \Gamma(y', s) - 3R$ and $|y_{n+1}| < \frac{1}{8}$. Then, $Q_{1/8}(p) \subset Q_{3/4}$ and

$$Q_{1/8}(p) \cap \{x_{n+1} = 0\} \subset \{x_n \leq \Gamma(x', t), x_{n+1} = 0\}.$$

Hence, $u \equiv 0$ on $Q_{1/8}(p) \cap \{x_{n+1} = 0\}$, and we can decompose $u = u_1 + u_2$, with $u_1 = u\chi_{\{x_{n+1}>0\}}$ and $u_2 = u\chi_{\{x_{n+1}<0\}}$.

Now, let $\tilde{u}_1$ and $\tilde{u}_2$ be the odd reflections of u_1 and u_2 across $\{x_{n+1} = 0\}$, and note that they satisfy $(\partial_t - \Delta)\tilde{u}_i = f_i$ in $Q_{1/8}(p)$, with f_i being the appropriate reflection of f. Therefore, by Theorem 5.2.5 again as in *Case 1*, $[u]_{C_p^{0,\gamma}(Q_\rho(p))} \leq C$.

Case 3. $R < \frac{1}{16}$, and $y_n > \Gamma(y', s) - 3R$ or $|y_{n+1}| \geq 2R$. Then, $Q_{2R}(p) \subset \Omega \cap Q_{3/4}$. Moreover, since $|y_{n+1}| \leq 3R$ and $|y_n - \Gamma(y', s)| \leq 3R$, $|u| \leq K'R^\gamma$ in $Q_{2R}(p)$. Therefore,

$$\begin{aligned}[u]_{C_p^{0,\gamma}(Q_R(p))} &\leq C\big(R^{-\gamma}\|u\|_{L^\infty(Q_{2R}(p))} + R^{(n+1)/(n+2)-\gamma}\|f\|_{L^{n+2}(Q_{2R}(p))}\big) \\ &\leq C\big(K' + R^{(n+1)/(n+2)-\gamma}\|f\|_{L^{n+2}(Q_1)}\big) \leq C'.\end{aligned}$$

Case 4. $R < \frac{1}{16}$, $y_n = \Gamma(y', s) - 3R$ and $|y_{n+1}| < 2R$. We proceed by an odd reflection as in *Case 2* and then use the estimates of *Case 3*. □

5.6.2 Special solution

Our goal now is to construct a special solution which is *almost homogeneous with parabolic homogeneity* $\frac{1}{2}$ (cf. Proposition 5.4.1 for the same result in *one-sided* Lipschitz domains).

Proposition 5.6.8. *Let $\varepsilon \in (0, \varepsilon_0)$, with ε_0 from Proposition 5.6.3. Then, there exist $\delta > 0$ and $C > 0$, only depending on ε and the dimension, such that the following holds.*

Let Ω be a parabolic slit domain in Q_1 in the sense of Definition 5.6.1 with Lipschitz constant δ. Let $\varphi_\pm$ be the parabolically homogeneous solutions introduced in Proposition 5.6.3, in a way that $\eta_\pm > \delta$ so that

$$Q_1 \setminus C^+_{\eta^+} \subset \Omega \subset Q_1 \setminus C^-_{\eta^-}.$$

Then, there exists $\varphi : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \varphi_t - \Delta\varphi = 0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial_\Gamma \Omega, \end{cases}$$

$\varphi \geq 0$, φ is even in x_{n+1}, $\|\varphi\|_{L^\infty(Q_1)} = 1$,

$$\frac{1}{C}\varphi_+ \leq \varphi \leq C\varphi_-,$$

and for all $0 < r_1 < r_2 \leq 1$,

$$\frac{\sup_{Q_{r_1}} \varphi}{\sup_{Q_{r_2}} \varphi} \geq \frac{1}{8}\left(\frac{r_1}{r_2}\right)^{1/2+\varepsilon}.$$

First, we prove the following almost positivity property for supersolutions to the heat equation.

Lemma 5.6.9. *Let $E \subset \{x_{n+1} = 0\} \cap Q_2$ be a closed set, and let w be a supersolution to*

$$\begin{cases} w_t - \Delta w & \geq & 0 & \text{in} & Q_2 \setminus E \\ w & = & 0 & \text{on} & E \\ w & \geq & 1 & \text{in} & Q_2 \cap \{|x_{n+1}| \geq \frac{1}{n+1}\} \\ w & \geq & -1 & \text{in} & Q_2. \end{cases}$$

Then,

$$w \geq 0 \quad \text{in } Q_1.$$

Proof. We will proceed by comparison with a barrier. Let $(y, s) \in Q_1$. If $y_{n+1} \geq \frac{1}{n+1}$, there is nothing to prove. Otherwise, consider the set

$$\Omega := B_1(y', y_n) \times \left(-\frac{1}{n+1}, \frac{1}{n+1}\right) \times (s-1, s) \setminus E.$$

By construction, $\Omega \subset Q_2 \setminus E$. Now, consider

$$v = w + 2|(x', x_n) - (y', y_n)|^2 + 4(s - t) - 2(n+1)|x_{n+1}|^2,$$

which by construction is also a supersolution for the heat equation in Ω. Moreover, on the parabolic boundary we can distinguish the following cases:

- When $t = s - 1$, $v \geq -1 + 4 - \frac{2}{n+1} \geq 0$.
- When $|(x', x_n) - (y', y_n)| = 1$, $v \geq -1 + 2 - \frac{2}{n+1} \geq 0$.
- When $|x_{n+1}| = \frac{1}{n+1}$, $v \geq 1 - \frac{2}{n+1} \geq 0$.
- On E, $v \geq w \geq 0$.

Therefore, by the comparison principle, $v(y, s) \geq 0$, and it follows that

$$w(y, s) = v(y, s) + 2(n+1)|y_{n+1}|^2 \geq 0.$$

□

As a consequence, we can deduce the following.

Lemma 5.6.10. *There exist $C, \varepsilon_0 > 0$, only depending on the dimension, such that for all $\varepsilon \in (0, \varepsilon_0)$,*

$$\varphi_+ \leq C\varphi_- \quad \text{in } Q_1,$$

where $\varphi_\pm$ are the parabolically homogeneous solutions introduced in Proposition 5.6.3.

Proof. Let

$$w := \frac{2}{m}\varphi_- - \frac{1}{2}\varphi_+$$

with m from Proposition 5.6.3.

Since $\{\varphi_- = 0\} \subset \{\varphi_+ = 0\}$, w is a supersolution for the heat equation in $Q_2 \setminus \{\varphi_- = 0\}$. Moreover, since $\|\varphi_+\|_{L^\infty(Q_2)} \leq 2^{1/2+\varepsilon}$ by homogeneity, $w \geq -1$ in Q_2, and $w \geq 1$ whenever $|x_{n+1}| \geq \frac{1}{n+1}$.

Hence, by Lemma 5.6.9, $w \geq 0$ in Q_1 and the conclusion follows. □

We will also need a Liouville theorem for slit domains:

Theorem 5.6.11. *Let $\alpha \in (0, \frac{1}{2})$, and let u be a solution to*

$$\begin{cases} (\partial_t - \Delta)u = 0 & \text{in } (\mathbb{R}^{n+1} \setminus \{x_n \le 0, x_{n+1} = 0\}) \times (-\infty, 0) \\ u = 0 & \text{on } \{x_n \le 0, x_{n+1} = 0\} \\ u(x', x_n, x_{n+1}, t) = u(x', x_n, -x_{n+1}, t) & \text{in } \mathbb{R}^{n+1} \times (-\infty, 0) \end{cases}$$

with the growth control

$$\|u\|_{L^\infty(Q_R)} \le C(1 + R^{1+\alpha}), \ \forall R \ge 1.$$

Then, $u = a\varphi_0$ for some $a \in \mathbb{R}$, where

$$\varphi_0(x, t) := \mathrm{Re}(\sqrt{x_n + i x_{n+1}}).$$

Proof. We will proceed as in [94, Theorem 4.11]. Let $\gamma \in (0, \frac{1}{2})$ such that $3\gamma > 1 + \alpha$, and let

$$u_r(x, t) := \frac{u(rx, r^2 t)}{r^{1+\alpha}}$$

for $r \ge 1$. Notice how u_r also satisfies the hypotheses.

Now, by Proposition 5.6.2,

$$[u]_{C^{0,\gamma}(Q_{2r})} = r^{1+\alpha-\gamma}[u_r]_{C^{0,\gamma}(Q_2)} \le Cr^{1+\alpha-\gamma}\|u_r\|_{L^\infty(Q_4)} \le Cr^{1+\alpha-\gamma}.$$

Then, given $h \in B_1$ such that $h_n = h_{n+1} = 0$ and $\tau \in (-1, 1)$, let

$$u^{(1)}(x, t) := u(x + h, t + \tau) - u(x, t)$$

and notice that it is also an even solution to the heat equation in the same domain. Hence,

$$\|u^{(1)}\|_{L^\infty(Q_r)} \le Cr^{1+\alpha-\gamma}.$$

Now we can repeat the procedure starting with $u^{(i)}$ instead of u to obtain that

$$\|u^{(2)}\|_{L^\infty(Q_r)} \le Cr^{1+\alpha-2\gamma}$$

and

$$\|u^{(3)}\|_{L^\infty(Q_r)} \le Cr^{1+\alpha-3\gamma},$$

where $u^{(i+1)} := u^{(i)}(\cdot + h, \cdot + \tau) - u^{(i)}$. Letting $r \to \infty$ in the last expression, we obtain that $u^{(3)}$ is identically zero, and therefore $u(\cdot, x_n, x_{n+1}, \cdot)$ is a third order polynomial, but from the growth condition on u we deduce that actually u is of the form

$$u(x, t) = \phi(x_n, x_{n+1}) + \psi(x_n, x_{n+1}) \cdot x' + \zeta(x_n, x_{n+1})t.$$

Since the domain is translation invariant in the x' and t directions, we deduce that ϕ, ψ, ζ are two-dimensional solutions to the heat equation that do not depend on time, i.e., harmonic functions in $\mathbb{R}^2 \setminus (-\infty, 0)$. Hence, they are of the form

$$\sum_{k=0}^{\infty} a_k \,\mathrm{Re}((x_n + i x_{n+1})^{k+1/2}) + \sum_{k=1}^{\infty} b_k \,\mathrm{Im}((x_n + i x_{n+1})^k).$$

Finally, from the growth and the even symmetry of u, we deduce that $\phi \equiv a_0 \,\mathrm{Re}((x_n + i x_{n+1})^{1/2})$ and $\psi \equiv \zeta \equiv 0$, as we wanted to prove. □

Remark 5.6.12. The assumption of u being even in the x_{n+1} direction is necessary to discard linear terms of the form $b_1 x_{n+1}$. This is motivated by the study of free boundary problems such as the parabolic Signorini problem, where the solution being even is a natural assumption (see for instance [69]).

Now, we are ready to replicate the strategy of Section 5.4. First, we construct solutions with a controlled growth.

Lemma 5.6.13. *Let $\varepsilon \in (0, \varepsilon_0)$, with ε_0 from Proposition 5.6.3. There exist $C > 0$, only depending on the dimension, and $\delta_1 \in (0, \varepsilon)$, only depending on the dimension and ε, such that the following holds.*

Let $R = 2^{1/\varepsilon}$ and let Ω be a parabolic slit domain in Q_R in the sense of Definition 5.6.1 with Lipschitz constant δ_1. Then, there exists $\varphi : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \varphi_t - \Delta\varphi &= 0 \quad \text{in} \quad \Omega \\ \varphi &= 0 \quad \text{on} \quad \partial_\Gamma\Omega, \end{cases}$$

$\varphi \geq 0$, φ is even in x_{n+1}, $\|\varphi\|_{L^\infty(Q_1)} = 1$, and

$$\frac{1}{C}\varphi_+ \leq \varphi \leq C\varphi_- \quad \text{in } Q_R.$$

In particular, $\|\varphi\|_{L^\infty(Q_r)} \leq Cr^{1/2-\varepsilon}$ for all $r \in [1, R]$.

Proof. First, by construction and Proposition 5.6.3, if δ_1 is small enough φ_+ is a subsolution and φ_- is a supersolution for the heat equation in Ω.

Now, from Lemma 5.6.10 and the different parabolic homogeneities of φ_+ and φ_-, we deduce that

$$\varphi_+ \leq CR^{2\varepsilon}\varphi_- = 4C\varphi_- \quad \text{in } Q_R.$$

Note that C does not depend on ε.

Therefore, by the comparison principle there exists $\tilde{\varphi} \geq 0$, a solution to the heat equation in Ω, vanishing on $\partial_\Gamma\Omega$, that satisfies

$$\varphi_+ \leq \tilde{\varphi} \leq 4C\varphi_- \quad \text{in } Q_R.$$

By Proposition 5.6.3, $1 \leq \|\tilde{\varphi}\|_{L^\infty(Q_1)} \leq 4C$, and therefore

$$\varphi := \frac{\tilde{\varphi}}{\|\tilde{\varphi}\|_{L^\infty(Q_1)}}$$

satisfies the first estimate. The second estimate follows directly from the parabolic scaling of φ_-. □

Now, the proof continues as in the one sided case.

Proof of Proposition 5.6.8. We follow the same strategy as in the proof of Proposition 5.4.1. First, Lemma 5.6.13 replaces Lemma 5.4.2. Then, an analogue to Lemma 5.4.3 can be proved by the same type of blow-up argument. To do so, we use Proposition 5.6.2 for the boundary regularity, and Theorem 5.6.11 to classify the blow-up limit.

The conclusion follows by an inductive argument as in Lemma 5.4.4, and combining the estimates as in the proof of Proposition 5.4.1. □

5.6.3 Expansion in slit domains

The following proposition follows the lines of Proposition 5.5.1, adapted to slit domains.

Proposition 5.6.14. *Let $\alpha \in (0, \frac{1}{2})$. There exists $\varepsilon_0 \in (0,1)$, only depending on α and the dimension such that the following holds.*

Let Ω be a parabolic slit domain in Q_1 in the sense of Definition 5.6.1 with Lipschitz constant ε_0. Let u be a solution to

$$\begin{cases} u_t - \Delta u &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial_\Gamma \Omega, \end{cases}$$

where u is even in x_{n+1}, $\|u\|_{L^\infty(Q_1)} \le 1$, and $\|f\|_{L^q(Q_1)} \le 1$ with $q = (n+3)/(1-\alpha)$.

Then, for each $r \in (0,1]$ there exists $K_r \in \mathbb{R}$ such that $|K_r| \le C$ and

$$\|u - K_r \varphi\|_{L^\infty(Q_r)} \le C r^{1+\alpha},$$

where φ is the special solution introduced in Proposition 5.6.8 and C depends only on α and the dimension.

Proof. We follow the same four steps as in Proposition 5.5.1.

Steps 1 and 4 are identical. Steps 2 and 3 have to be modified in the same way. We will only write the modified Step 2.

Modified Step 2. We prove that $w_m \to w$ locally uniformly along a subsequence, where w is a solution to

$$\begin{cases} w_t - \Delta w &= 0 \quad \text{in} \quad \mathbb{R}^{n+2} \setminus \{x_n \le 0, x_{n+1} = 0\} \\ w &= 0 \quad \text{on} \quad \{x_n \le 0, x_{n+1} = 0\}. \end{cases} \tag{5.7}$$

Then, by the construction of w_m (omitting the dependence of f on j_m),

$$|(\partial_t - \Delta) w_m| \le \frac{2\rho_m^{1-\alpha}}{\theta(\rho_m)} f(\rho_m x, \rho_m^2 t),$$

and estimating the right-hand side

$$\left\| \frac{2\rho_m^{1-\alpha}}{\theta(\rho_m)} f(\rho_m x, \rho_m^2 t) \right\|_{L^q(Q_1)} = \frac{2\rho_m^{1-\alpha}}{\theta(\rho_m)} \rho_m^{-(n+3)/q} \|f\|_{L^q(Q_{\rho_m})} \le \frac{2\|f\|_{L^q(Q_1)}}{\theta(\rho_m)} \le \frac{2}{\theta(\rho_m)},$$

where we used that $(n+3)/q = 1 - \alpha$.

Moreover, $w_m = 0$ on the appropriate rescaling of $\partial_\Gamma \Omega_{j_m}$, and, for every $R \ge 1$, $\|w_m\|_{L^\infty(Q_R)} \le CR^{1+\alpha}$ for sufficiently large m. Hence, by Proposition 5.6.2,

$$\|w_m\|_{C_p^{0,1/4}(Q_R)} \le C(R),$$

uniformly in m, for m large enough. Then, by Arzelà-Ascoli and Proposition 5.2.7, we obtain that

$$w_m \to w \in C(\mathbb{R}^{n+2}),$$

locally uniformly along a subsequence, where w is a viscosity solution to (5.7) and $\|w\|_{L^\infty(Q_R)} \le CR^{1+\alpha}$ for all $R \ge 1$. Therefore, by Theorem 5.6.11 and the fact that $\|w\|_{L^\infty(Q_1)} = 1$, $w = \varphi_0$. In the rest of the proof, φ_0 plays the role of $(x_n)_+$ in the proof of Proposition 5.5.1.

□

5.6.4 The Boundary Harnack in slit domains

Proof of Theorem 5.1.6. The strategy of the proof is the same as in Theorem 5.1.2. Let $\alpha = 1 - \frac{n+3}{q}$, and $\varepsilon \in (0, \varepsilon_0)$ (with ε_0 from Proposition 5.6.8) such that $\gamma \leq \frac{1}{2} + \alpha - 17\varepsilon$.

Let $R_0 \in (0, \frac{1}{48})$ to be chosen later, $p = (y', y_n, s)$ and $\rho \in (0, R_0)$ such that $Q_{2\rho}(p) \subset \Omega \cap Q_{5/8}$. Assume without loss of generality that $y_{n+1} \geq 0$, and let

$$R := \max\left\{\rho, \frac{y_{n+1}}{8}, \frac{|y_n - \Gamma(y', s)|}{8}\right\}.$$

Then, we distinguish four cases (cf. Proposition 5.6.2).

Case 1. $R \geq R_0$, and $y_n > \Gamma(y', s) - 8R$ or $y_{n+1} \geq 2R$. Then, $Q_{2R}(p) \subset \Omega \cap Q_{3/4}$. Now, let us consider two subcases:

- If $y_{n+1} < 2R$, $y_n = \Gamma(y', s) + 8R$. Therefore, for all $(x, t) \in Q_R(p)$,

$$x_n \geq \Gamma(x', t) + (7 - 2c_0)R > \Gamma(x', t) + 6R.$$

- If $y_{n+1} \geq 2R$, since $y_n \geq \Gamma(y', s) - 8R$, for all $(x, t) \in Q_R(p)$,

$$x_n \geq \Gamma(x', t) - (9 + 2c_0)R > \Gamma(x', t) - 10R \geq \Gamma(x', t) - 10x_{n+1}.$$

In both cases it holds that, for all $(x, t) \in Q_R(p)$,

$$x_n - \Gamma(x', t) + 10|x_{n+1}| > 0,$$

and

$$|x_n - \Gamma(x', t)| + |x_{n+1}| > 2R.$$

Hence, by Proposition 5.6.7, $v \geq cm > 0$ in $Q_R(p)$. Furthermore, by Proposition 5.6.2, $\|u\|_{C_p^{0,\gamma}(Q_R(p))} \leq C$ and $\|v\|_{C_p^{0,\gamma}(Q_R(p))} \leq C$. Therefore,

$$\left\|\frac{u}{v}\right\|_{C_p^{0,\gamma}(Q_\rho(p))} \leq \frac{\|u\|_{C_p^{0,\gamma}(Q_\rho(p))}\|v\|_{L^\infty(Q_\rho(p))} + \|u\|_{L^\infty(Q_\rho(p))}\|v\|_{C_p^{0,\gamma}(Q_\rho(p))}}{\inf\limits_{Q_\rho(p)} v^2} \leq Cm^{-2}.$$

Case 2. $R \geq R_0$, $y_n = \Gamma(y', s) - 8R$ and $y_{n+1} < 2R_0$. Let

$$E := Q^+_{1/8}(y', y_n, 0, s) := Q_{1/8}(y', y_n, 0, s) \cap \{x_{n+1} > 0\} \subset Q_{3/4}.$$

Then,

$$\overline{E} \cap \{x_{n+1} = 0\} \subset \{x_n \leq \Gamma(x', t),\ x_{n+1} = 0\}.$$

Moreover, $\rho < R_0 < \frac{1}{48}$, and hence $Q_\rho(p) \subset Q^+_{1/16}(y', y_n, 0, s)$.

We will apply Theorem 5.1.2 with the following functions defined in Q_1^+:

$$\tilde{u}(x, t) := u\left((y', y_n, 0) + \frac{x}{8}, s + \frac{t}{64}\right) \quad \text{and} \quad \tilde{v}(x, t) := \frac{v\left((y', y_n, 0) + \frac{x}{8}, s + \frac{t}{64}\right)}{\|v\|_{L^\infty(Q^+_{1/8}(y', y_n, 0, s))}}.$$

First, $\|\tilde{u}\|_{L^\infty(Q_1^+)} \leq 1$, $v > 0$ and $\|\tilde{v}\|_{L^\infty(Q_1^+)} = 1$, and the domain is a half-space so it has Lipschitz constant 0. Then, about the right-hand side of the equation,

$$\tilde{u}_t - \Delta \tilde{u} = \frac{1}{64} f_1 \left((y', y_n, 0) + \frac{x}{8}, s + \frac{t}{64} \right),$$

and hence

$$\|\tilde{u}_t - \Delta \tilde{u}\|_{L^q(Q_1^+)} \leq \frac{8^{(n+3)/q}}{64} \|f\|_{L^q(Q_1)} \leq 1.$$

Moreover,

$$\tilde{v}\left(\frac{e_{n+1}}{2}, -\frac{3}{4}\right) = \frac{v((y', y_n, \frac{1}{16}), -\frac{3}{256})}{\|v\|_{L^\infty(Q^+_{1/8}(y', y_n, 0, s))}} \geq \frac{c_2 m}{\|v\|_{L^\infty(Q^+_{1/8}(y', y_n, 0, s))}} > 0,$$

by Proposition 5.6.7.

Therefore, we can apply Theorem 5.1.2 to obtain

$$\left\| \frac{\tilde{u}}{\tilde{v}} \right\|_{C_p^{0,\gamma}(Q^+_{1/2})} \leq C \|v\|^2_{L^\infty(Q^+_{1/8}(y', y_n, 0, s))} m^{-2},$$

and hence

$$\left\| \frac{u}{v} \right\|_{C_p^{0,\gamma}(Q_\rho(p))} \leq C \|v\|_{L^\infty(Q^+_{1/16}(y', y_n, 0, s))} m^{-2} \leq C m^{-2}.$$

Case 3. $R < R_0$, and $y_n > \Gamma(y', s) - 8R$ or $y_{n+1} \geq 2R$. Then, $Q_{2R}(p) \subset \Omega \cap Q_{3/4}$. Analogously to *Case 1*, we can apply Proposition 5.6.7 to obtain $v \geq cmR^{1/2+\varepsilon}$ in $Q_R(p)$. Then, the right-hand side of the equation for u can be estimated in L^{n+2} as

$$\begin{aligned} \|f_1\|_{L^{n+2}(Q_{2R}(p))} &\leq R^{(n+3)(1/(n+2) - 1/q)} \|f_1\|_{L^q(Q_{2R}(p))} \\ &\leq R^{1/(n+2)+\alpha} \|f_1\|_{L^q(Q_1)} \leq R^{1/(n+2)+\alpha}, \end{aligned}$$

and analogously $\|f_2\|_{L^{n+2}(Q_{2R}(p))} \leq c_0 m R^{1/(n+2)+\alpha}$.

Now, let φ be the special solution introduced in Proposition 5.6.8, centered at $(y', \Gamma(y', s), 0, s)$. Then, $w_1 = u - K_u \varphi$ and $w_2 = v - K_v \varphi$ satisfy

$$\|w_1\|_{L^\infty(Q_{2R}(p))} \leq C R^{1+\alpha} \quad \text{and} \quad \|w_2\|_{L^\infty(Q_{2R}(p))} \leq C R^{1+\alpha}$$

by a translation of Proposition 5.6.14.

Finally, we proceed as in the proof of Theorem 5.1.2. By the interior estimates in Theorem 5.2.5, and the growth of v and φ (see Propositions 5.6.7 and 5.6.8), and using that $u = w_1 + K_u \varphi$, we estimate

$$\begin{aligned} [w_1/v]_{C_p^{0,\gamma}(Q_R(p))} &\leq \frac{[w_1]_{C_p^{0,\gamma}(Q_R(p))} \|v\|_{L^\infty(Q_R(p))} + \|w_1\|_{L^\infty(Q_R(p))} [v]_{C_p^{0,\gamma}(Q_R(p))}}{\inf\limits_{Q_R(p)} v^2} \\ &\leq C \frac{R^{1+\alpha-\gamma} R^{1/2-\varepsilon} + R^{1+\alpha} R^{1/2-\gamma-\varepsilon}}{m^2 R^{2(1/2+\varepsilon)}} \leq C m^{-2} \end{aligned}$$

and

$$\begin{aligned} [\varphi/v]_{C_p^{0,\gamma}(Q_R(p))} &\leq \frac{[\varphi]_{C_p^{0,\gamma}} \|w_2\|_{L^\infty} + [w_2]_{C_p^{0,\gamma}} \|\varphi\|_{L^\infty}}{\inf(v/\varphi)^2 \inf \varphi^2} \\ &\leq C \frac{R^{1/2-\varepsilon-\gamma} R^{1+\alpha} + R^{1+\alpha-\gamma} R^{1/2-\varepsilon}}{(m R^{2\varepsilon})^2 (R^{1/2+\varepsilon})^2} \leq 2C m^{-2}, \end{aligned}$$

where we omitted the domain to improve readability, and therefore

$$[u/v]_{C_p^{0,\gamma}(Q_R(p))} \le [w_1/v]_{C_p^{0,\gamma}(Q_R(p))} + |K_u|[\varphi/v]_{C_p^{0,\gamma}(Q_R(p))} \le Cm^{-2}.$$

Case 4. $R < R_0$, $y_n = \Gamma(y', s) - 8R$ and $y_{n+1} < 2R$. Let

$$E := Q_{6R}^+(y', y_n, 0, s) \subset Q_{3/4}.$$

Then,

$$\overline{E} \cap \{x_{n+1} = 0\} \subset \{x_n \le \Gamma(x', t)\}.$$

Moreover, $y_{n+1} + \rho < 3R$, and then $Q_\rho(p) \subset Q_{3R}^+(y', y_n, 0, s)$. As in *Case 3*, we can apply Proposition 5.6.14 to obtain $w_1 = u - K_u\varphi$ and $w_2 = v - K_v\varphi$ satisfying

$$\|w_1\|_{L^\infty(E)} \le CR^{1+\alpha} \quad \text{and} \quad \|w_2\|_{L^\infty(E)} \le CR^{1+\alpha}.$$

Assume without loss of generality that $C \ge 36$. Now, let $x_0 = (y', y_n, 0)$ and let

$$\tilde{w}_1(x,t) := \frac{w_1(x_0 + 6Rx, s + 36R^2 t)}{CR^{1+\alpha}} \quad \text{and} \quad \tilde{v}(x,t) := \frac{v(x_0 + 6Rx, s + 36R^2 t)}{\|v\|_{L^\infty(E)}}.$$

Since $y_n = \Gamma(y', s) - 8R$, $(y', y_n, 3R, s) \in E$ satisfies

$$y_n - \Gamma(y', s) + 10 \cdot 3R \ge 0,$$

and by Proposition 5.6.7, $v(y', y_n, 3R, s) \ge cmR^{1/2+\varepsilon}$, and thus $\|v\|_{L^\infty(E)} \ge cmR^{1/2+\varepsilon}$. Therefore, $\|w_1\|_{L^\infty(Q_1^+)} \le 1$, $\|v\|_{L^\infty(Q_1^+)} = 1$, and the right-hand sides satisfy

$$\|(\partial_t - \Delta)\tilde{w}_1\|_{L^q(Q_1^+)} = \frac{36R^2}{CR^{1+\alpha}}(6R)^{-(n+3)/q}\|f_1\|_{L^q(E)} \le 1$$

and

$$\begin{aligned}\|(\partial_t - \Delta)\tilde{v}\|_{L^q(Q_1^+)} &= \frac{36R^2}{\|v\|_{L^\infty(E)}}(6R)^{-(n+3)/q}\|f_2\|_{L^q(E)} \\ &\le \frac{36R^2(6R)^{\alpha-1}}{cmR^{1/2+\varepsilon}}c_0 m \le c_0(cmR^{2\varepsilon}),\end{aligned}$$

provided that $R \le R_0$ is small enough, where we also used that $(n+3)/q = 1 - \alpha$. Finally, by Proposition 5.6.7, $v(x_0 + 3Re_n, s - 27R^2) \ge cmR^{1/2+\varepsilon}$ which implies $\tilde{v}(\frac{e_n}{2}, -\frac{3}{4}) \ge cmR^{2\varepsilon}$, and also $\|v\|_{L^\infty(E)} \le CR^{1/2-\varepsilon}$. Thus, by Theorem 5.1.2,

$$[\tilde{w}_1/\tilde{v}]_{C_p^{0,\gamma}(Q_1^+)} \le C(m)R^{-4\varepsilon},$$

and undoing the scaling,

$$[w_1/v]_{C_p^{0,\gamma}(E)} \le \frac{CR^{1+\alpha}}{\|v\|_{L^\infty(E)}}R^{-\gamma}[\tilde{w}_1/\tilde{v}]_{C_p^{0,\gamma}(Q_1^+)} \le C(m)R^{1/2+\alpha-\gamma-5\varepsilon} \le C(m).$$

Consider now

$$\tilde{w}_2(x,t) := \frac{w_2(x_0 + 6Rx, s + 36R^2 t)}{CR^{1+\alpha}} \quad \text{and} \quad \tilde{\varphi}(x,t) := \frac{\varphi(x_0 + 6Rx, s + 36R^2 t)}{\|\varphi\|_{L^\infty(E)}}.$$

By the parabolic homogeneity of φ, we get $\tilde{\varphi}(\frac{e_n}{2}, -\frac{3}{4}) \geq cR^{2\varepsilon}$; see Proposition 5.6.8. We also have that

$$\|(\partial_t - \Delta)\tilde{w}_2\|_{L^q(Q_1^+)} \leq c_0 m$$

by the same reasoning as with $\tilde{w}_1$.

By Theorem 5.1.2,

$$[\tilde{w}_2/\tilde{\varphi}]_{C_p^{0,\gamma}(Q_1^+)} \leq C(m)R^{-4\varepsilon},$$

and undoing the scaling,

$$[w_2/\varphi]_{C_p^{0,\gamma}(E)} \leq \frac{CR^{1+\alpha}}{\|\varphi\|_{L^\infty(E)}} R^{-\gamma}[\tilde{w}_2/\tilde{\varphi}]_{C_p^{0,\gamma}(Q_1^+)} \leq C(m)R^{1/2+\alpha-\gamma-5\varepsilon}.$$

On the other hand, by Corollary 5.1.5,

$$c(m)R^{4\varepsilon} \leq \frac{\tilde{v}}{\tilde{\varphi}} \leq C(m)R^{-4\varepsilon} \text{ in } Q_1^+,$$

which after undoing the scaling (by Proposition 5.6.7) becomes

$$c(m)R^{6\varepsilon} \leq \frac{v}{\varphi} \leq C(m)R^{-6\varepsilon} \text{ in } E.$$

Hence, we can compute

$$[\varphi/v]_{C_p^{0,\gamma}(E)} \leq \frac{[v/\varphi]_{C_p^{0,\gamma}(E)}}{\inf_E (v/\varphi)^2} = \frac{[w_2/\varphi]_{C_p^{0,\gamma}(E)}}{\inf_E (v/\varphi)^2}$$

and applying the previous estimates

$$[\varphi/v]_{C_p^{0,\gamma}(E)} \leq C(m)R^{1/2+\alpha-\gamma-17\varepsilon} \leq C(m).$$

Finally, as in *Case 3*,

$$[u/v]_{C_p^{0,\gamma}(Q_R(p))} \leq [w_1/v]_{C_p^{0,\gamma}(Q_R(p))} + |K_u|[\varphi/v]_{C_p^{0,\gamma}(Q_R(p))} \leq C(m),$$

as we wanted to prove. □

5.7 Applications to free boundary problems

5.7.1 $C^{1,\alpha}$ free boundary regularity for the parabolic obstacle problem

The argument in this proof is standard, we write it for the sake of completeness.

Proof of Corollary 5.1.8. Let $e \in \mathbb{R}^{n+1}$ be a unit vector. Then, since $u \in C^1$, u_e is a solution to

$$\begin{cases} \partial_t u_e - \Delta u_e &= f_e \quad \text{in} \quad \{u > 0\} \\ u_e &= 0 \quad \text{on} \quad \partial\{u > 0\}. \end{cases}$$

Let now $r > 0$ to be chosen later, and define the functions

$$w_1 := \frac{u_e(rx, r^2t)}{\max\{\|u_e\|_{L^\infty(Q_r)}, Cr\}} \quad \text{and} \quad w_2 := \frac{u_n(rx, r^2t)}{\|u_n\|_{L^\infty(Q_r)}}.$$

Then, $\|w_1\|_{L^\infty(Q_1)} \le 1$, $\|w_2\|_{L^\infty(Q_1)} = 1$ and $w_2 > 0$. Furthermore, $\|u_n\|_{L^\infty(Q_r)} \le Cr$ by the $C^{1,1}_x$ regularity of u, and it follows that

$$w_2\left(\frac{e_n}{2}, -\frac{3}{4}\right) \ge \frac{cd(re_n/2, -3r^2/4)}{Cr} \ge \frac{c}{4C},$$

using that $d(re_n/2, -3r^2/4) \ge r/4$ if the Lipschitz constant of the domain is small enough. Finally,

$$\|(\partial_t - \Delta)w_1\|_{L^q(Q_1)} \le \frac{r^{2-(n+2)/q}\|\nabla f\|_{L^q(Q_r)}}{Cr} \le \frac{\|\nabla f\|_{L^q(Q_1)}}{C} r^{1-(n+2)/q}$$

and

$$\|(\partial_t - \Delta)w_2\|_{L^q(Q_1)} \le \frac{r^{2-(n+2)/q}\|\nabla f\|_{L^q(Q_r)}}{u_n(re_n/2, -r^2/2)} \le \frac{4\|\nabla f\|_{L^q(Q_1)}}{c} r^{1-(n+2)/q}.$$

Therefore, choosing $r > 0$ small enough (independent of e) we can apply Theorem 5.1.2 to w_1 and w_2 and obtain that $w_1/w_2 \in C^{0,\alpha}_p(Q_{1/2} \cap \{w_2 > 0\})$. Now letting $e = e_i$ for all vectors of the coordinate basis, we obtain that

$$\frac{(\nabla u, u_t)}{u_n} \in C^{0,\alpha}_p(Q_{r/2} \cap \{u > 0\}) \subset C^{0,\alpha/2}(Q_{r/2} \cap \{u > 0\}).$$

Notice also that the modulus of this function is bounded below by 1.

Now, the normal vector to the level sets $\{u = t\}$ for $t > 0$ can be written as

$$\hat{n} = \frac{(\nabla u, u_t)}{\sqrt{|\nabla u|^2 + u_t^2}} = \frac{\frac{(\nabla u, u_t)}{u_n}}{\left|\frac{(\nabla u, u_t)}{u_n}\right|} \in C^{0,\alpha/2}(Q_{r/2} \cap \{u > 0\}),$$

hence $\{u = t\}$ is a $C^{1,\alpha/2}$ hypersurface, and taking the limit as $t \downarrow 0$ (uniformly because $u \in C^1$), we obtain that $\partial\{u > 0\}$ is $C^{1,\alpha/2}$ as well. □

5.7.2 $C^{1,\alpha}$ free boundary regularity for the parabolic Signorini problem

In the case of slit domains, the proof has to be slightly modified to account for the different scaling of the solution.

Proof of Corollary 5.1.9. Let $e \in \mathbb{R}^{n+2}$ be a unit vector. Then, since $u \in C^1$, u_e is a solution to

$$\begin{cases} \partial_t u_e - \Delta u_e &= f_e \quad \text{in} \quad Q_1 \setminus \Lambda(u) \\ u_e &= 0 \quad \text{on} \quad \Lambda(u). \end{cases}$$

Let now $r > 0$ to be chosen later, and define the functions

$$w_1 := \frac{u_e(rx, r^2t)}{\max\{\|u_e\|_{L^\infty(Q_r)}, Cr^{1/2}\}} \quad \text{and} \quad w_2 := \frac{u_n(rx, r^2t)}{\|u_n\|_{L^\infty(Q_r)}}.$$

Then, $\|w_1\|_{L^\infty(Q_1)} \leq 1$, $\|w_2\|_{L^\infty(Q_1)} = 1$ and $w_2 > 0$. Furthermore, $\|u_n\|_{L^\infty(Q_r)} \leq Cr^{1/2}$ by the $C_x^{3/2}$ regularity of u, and it follows that

$$w_2\left(\frac{e_n}{2}, -\frac{3}{4}\right) \geq \frac{cd(re_n/2, -3r^2/4)^{1/2}}{Cr^{1/2}} \geq \frac{c}{2C},$$

using that $d(re_n/2, -3r^2/4) \geq r/4$ if the Lipschitz constant of the domain is small enough. Finally,

$$\|(\partial_t - \Delta)w_1\|_{L^q(Q_1)} \leq \frac{r^{2-(n+3)/q}\|\nabla f\|_{L^q(Q_r)}}{Cr^{1/2}} \leq \frac{\|\nabla f\|}{C} r^{3/2-(n+3)/q}$$

and

$$\|(\partial_t - \Delta)w_2\|_{L^q(Q_1)} \leq \frac{r^{2-(n+3)/q}\|\nabla f\|_{L^q(Q_r)}}{u_n(re_n/2, -r^2/2)} \leq \frac{4\|\nabla f\|}{c} r^{3/2-(n+3)/q}.$$

Therefore, choosing $r > 0$ small enough (independent of e) we can apply Theorem 5.1.6 to w_1 and w_2 and obtain that $w_1/w_2 \in C_p^{0,\alpha}(Q_{1/2} \cap \{w_2 > 0\})$. From here the argument goes on as in the proof of Corollary 5.1.8. □

5.8 The elliptic boundary Harnack with right-hand side

Applying similar reasoning as in Sections 5.3, 5.4, and 5.5, but using elliptic instead of parabolic theory, one can arrive to the following result, that generalizes the right-hand sides considered in [3] and [169] for non-divergence operators.

It is noteworthy that even for the Laplacian, this is the first optimal regularity result for quotients of solutions in domains that are less regular than C^1.

First, let us define a Lipschitz domain in the elliptic setting.

Definition 5.8.1. We say Ω is a Lipschitz domain in B_R with Lipschitz constant L if Ω is the epigraph of a Lipschitz function $\Gamma : B'_R \to \mathbb{R}$, with $\Gamma(0,0) = 0$:

$$\Omega = \left\{(x', x_n) \in B'_R \times (-R, R) \mid x_n > \Gamma(x')\right\}, \quad \|\Gamma\|_{C^{0,1}} \leq L.$$

In this context, we will denote the lateral boundary

$$\partial_\Gamma \Omega := \left\{(x, x_n) \in B'_R \times (-R, R) \mid x_n = \Gamma(x')\right\},$$

which is a subset of the topological boundary of Ω.

Remark 5.8.2. To extend the concept of regularized distance to *one-sided* elliptic Lipschitz domains, a simple approach is to establish a correspondence between elliptic domains and time-independent parabolic domains by adding a dummy variable.

Finally, it is worth highlighting that the key difference in the proof is the change in scaling between the parabolic ABP estimate, Theorem 5.2.4, and its elliptic counterpart, which we state below.

Theorem 5.8.3 ([37, Theorem 3.2]). *Let $\mathcal{L}$ be a non-divergence form operator as in (5.1) and let $u \in W^{2,n}_{\text{loc}}$ be a solution to $\mathcal{L}u = f$ in B_r, with $f \in L^n(B_r)$.*

Then,

$$\sup_{B_r} u \leq \sup_{\partial B_r} u^+ + Cr\|f\|_{L^n(B_r)},$$

where C depends only on the dimension and the ellipticity constants.

Now we are ready to state our main elliptic result.

Theorem 5.8.4. *Let $0 < \gamma < \alpha < \alpha_0$, $m \in (0,1]$ and let $\mathcal{L}$ be a non-divergence form operator as in (5.1). There exists $c_0 \in (0,1)$, only depending on α, γ, the dimension and the ellipticity constants, such that the following holds.*

Let Ω be a Lipschitz domain in B_1 in the sense of Definition 5.8.1 with Lipschitz constant $L \leq c_0$. Let u and v be solutions to

$$\begin{cases} \mathcal{L}u &= f_1 \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial_\Gamma \Omega \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}v &= f_2 \quad \text{in} \quad \Omega \\ v &= 0 \quad \text{on} \quad \partial_\Gamma \Omega, \end{cases}$$

and assume that $\|u\|_{L^\infty(B_1)} \leq 1$, $\|v\|_{L^\infty(B_1)} = 1$, $v > 0$, $v\left(\frac{e_n}{2}\right) \geq m$, and that $f_i = g_i + h_i$ with

$$\|d^{1-\alpha} g_1\|_{L^\infty(B_1)} + \|d^{-\alpha} h_1\|_{L^n(B_1)} \leq 1$$

and

$$\|d^{1-\alpha} g_2\|_{L^\infty(B_1)} + \|d^{-\alpha} h_2\|_{L^n(B_1)} \leq c_0 m,$$

where d is the regularized distance introduced in Remark 5.8.5.

Then,

$$\left\|\frac{u}{v}\right\|_{C^{0,\gamma}(\Omega \cap B_{1/2})} \leq C,$$

where C depends only on m, α, γ, the dimension and the ellipticity constants.

Remark 5.8.5. The function space considered for the right-hand side is the most general allowed by our proof. Notice how the weighted L^{n+1} norm of the parabolic result translates into a weighted L^n norm in the elliptic setting, due to the different scaling of the ABP estimate.

By a similar argument to Proposition 5.10.2, Theorem 5.8.4 allows for $f_i \in L^q$ with $q = n/(1-\alpha)$, generalizing [169]. It also allows for $|f_i| \leq c_0 m d^{\alpha-1}$ as in [3].

Proof. It follows from the proof of Theorem 5.1.2 and the previous lemmas, using Theorem 5.8.3 instead of Theorem 5.2.4. Notice that if we consider the solution to an elliptic problem as a stationary solution for the parabolic problem and try to apply Theorem 5.1.2 directly, we obtain a weaker result. □

If we can interchange the roles of u and v, we derive a corollary in a similar manner to the parabolic case.

Corollary 5.8.6. *Let $\alpha \in (0, \alpha_0)$, $m \in (0,1]$ and let $\mathcal{L}$ be a non-divergence form operator as in (5.1). There exists $c_0 \in (0,1)$, only depending on α, the dimension and the ellipticity constants, such that the following holds.*

Let Ω be a Lipschitz domain in B_1 in the sense of Definition 5.8.1 with Lipschitz constant $L \le c_0$. Let u and v be positive solutions to

$$\begin{cases} \mathcal{L}u &= f_1 \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial_\Gamma\Omega \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}v &= f_2 \quad \text{in} \quad \Omega \\ v &= 0 \quad \text{on} \quad \partial_\Gamma\Omega, \end{cases}$$

and assume that $\|u\|_{L^\infty(B_1)} = \|v\|_{L^\infty(B_1)} = 1$, $v\left(\frac{e_n}{2}\right) \ge m$, $u\left(\frac{e_n}{2}\right) \ge m$, and that $f_i = g_i + h_i$ with

$$\|d^{1-\alpha} g_i\|_{L^\infty(B_1)} + \|d^{-\alpha} h_i\|_{L^n(B_1)} \le c_0 m,$$

where d is the regularized distance introduced in Remark 5.8.5.

Then,

$$\frac{1}{C} \le \frac{u}{v} \le C \quad \text{in } \Omega \cap B_{1/2},$$

where C depends only on m, α, the dimension and the ellipticity constants.

Remark 5.8.7. The analogous theorems hold for the right notion of *elliptic slit domains* with a right-hand side with the same conditions as in Theorem 5.8.4.

5.9 Proof of Corollary 5.1.11

Using the boundary Harnack, we can combine it with the Hopf lemma to get a Hopf-type estimate for solutions of parabolic equations with a right-hand side. Let us start by defining Dini continuity and the interior $C^{1,\mathrm{Dini}}$ condition.

Definition 5.9.1. We say $\omega : [0,\infty) \to [0,\infty)$ is a Dini modulus of continuity if it is nondecreasing and there exists $r_0 > 0$ such that

$$\int_0^{r_0} \omega(r) \frac{\mathrm{d}r}{r} < \infty.$$

Definition 5.9.2. Given Ω a parabolic Lipschitz domain in Q_R, we say Ω satisfies the interior parabolic $C^{1,\mathrm{Dini}}$ condition at 0 if (possibly after a rotation) there exists $r_0 > 0$ and a Dini modulus of continuity ω such that

$$\{(x', x_n, t) \in Q_{r_0} \mid x_n > (|x'| + |t|^{1/2})\omega(|x'| + |t|^{1/2})\} \subset \Omega.$$

Our starting point will be the following boundary point lemma for parabolic $\mathcal{C}^{1,\mathrm{Dini}}$ domains. We were surprised to not find it in the literature, so we provide it here for completeness. The proof follows the steps in [147] and relies on a standard iteration scheme.

Theorem 5.9.3. *Let $\mathcal{L}$ be a non-divergence form operator as in (5.1). Let Ω be a parabolic Lipschitz domain in Q_1 in the sense of Definition 5.2.2, and assume that it satisfies the interior $C^{1,\mathrm{Dini}}$ condition at 0.*

Let u be a positive solution to

$$\begin{cases} u_t - \mathcal{L}u &= 0 \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial_\Gamma\Omega, \end{cases}$$

and assume that $u\left(\frac{e_n}{2}, -\frac{3}{4}\right) = 1$*. Then, for all* $r \in (0, \delta)$*,*

$$u(re_n, 0) \geq cr,$$

where $c > 0$ *and* δ *depend only on the dimension, the ellipticity constants, and the modulus of continuity of the domain.*

We start with an auxiliary lemma for sequences.

Lemma 5.9.4. *Let* $C > 0$*, and let* $\{a_k\}$ *and* $\{w_k\}$ *be sequences of positive real numbers such that* $a_{k+2} = a_{k+1} - Cw_k a_k$*. Then, if*

$$\sum w_k \leq \frac{1}{2C}\left(2 + \frac{a_1}{a_2}\right)^{-1},$$

then $a_k > a_2/6$ *for all* $k \geq 2$*.*

Proof. First, notice that we may assume $C = 1$ without loss of generality. Then,

$$a_3 = a_2 - w_1 a_1 \geq a_2\left(1 - w_1\frac{a_1}{a_2}\right) > a_2\left(1 - \frac{1}{2}\right) = \frac{a_2}{2}.$$

Furthermore, if we assume that $2a_{k+1} > a_k$,

$$a_{k+2} = a_{k+1}\left(1 - w_k\frac{a_k}{a_{k+1}}\right) > a_{k+1}\left(1 - \frac{a_k}{4a_{k+1}}\right) > \frac{a_{k+1}}{2}.$$

Hence, by induction we see that $2a_{k+1} > a_k$ for all $k \geq 2$.

Now, by iterating the recurrence we also have, for $k \geq 3$,

$$a_k > a_3(1 - 2w_3)(1 - 2w_4) \cdot \ldots \cdot (1 - 2w_{k-1}),$$

and since $2w_k < 1/2$ for all k,

$$a_k > a_3 e^{-4w_3} e^{-4w_4} \cdot \ldots \cdot e^{-4w_{k-1}} > a_3 e^{-4\sum w_j} > \frac{a_3}{e} > \frac{a_2}{6},$$

where we used that $e^{-2x} < (1 - x)$ for all $x \in (0, 1/2)$. □

Now we are ready to prove the main statement of the section.

Proof of Theorem 5.9.3. First, from the interior Dini condition we get that there exists $r_0 > 0$ such that

$$\omega(r_0) \leq c_0 \quad \text{and} \quad \int_0^{r_0} \omega(r)\frac{\mathrm{d}r}{r} \leq c_0,$$

where $c_0 > 0$ is a small constant to be chosen later. After scaling, and using the parabolic Harnack inequality, we may assume $r_0 = 1$ and $u(e_n/2, -1/2) = 1$.

Then, we denote

$$\Omega_r^+ := \Omega \cap Q_r \cap \{x_n > 0\}.$$

By the Hopf lemma for flat boundaries applied to the set $\{x_n > \omega(1)\}$, we obtain $u \geq c_1(x_n - \omega(1))$ in $Q_{1/2} \cap \{x_n > \omega(1)\}$, and using that $u \geq 0$,

$$u \geq c_1 x_n - c_1\omega(1) \quad \text{in } \Omega_{1/2}^+.$$

Let $a_1 := c_1$ and $b_1 := c_1\omega(1)$. We will prove by induction that

$$u \geq a_k x_n - b_k \quad \text{in } \Omega^+_{2^{-k}},$$

for some positive sequences a_k and b_k satisfying the recurrence relations

$$\begin{cases} a_{k+1} &= a_k - 2^k C b_k \\ b_{k+1} &= 2^{-k}\omega(2^{-k})a_k. \end{cases}$$

Indeed, assume it true for a certain k and let $v = u - a_k x_n$, which is $\mathcal{L}$-caloric in $\Omega^+_{2^{-k}}$. On the one hand, $v \geq -b_k$ by induction hypothesis. On the other hand, $v \geq -a_k x_n$ because $u \geq 0$.

Now, let us estimate v in $\Omega^+_{2^{-k-1}}$ from below. To do so, let us define w_1 and w_2 as

$$\begin{cases} (\partial_t - \mathcal{L})w_1 &= 0 \quad \text{in } Q_{2^{-k}} \cap \{x_n > 0\} \\ w_1 &= -b_k \quad \text{on } \partial_p Q_{2^{-k}} \cap \{x_n > 0\} \\ w_1 &= 0 \quad \text{on } \{x_n = 0\}, \end{cases}$$

and

$$\begin{cases} (\partial_t - \mathcal{L})w_2 &= 0 \quad \text{in } \Omega \cap Q_{2^{-k}} \\ w_2 &= 0 \quad \text{on } \partial_p(\Omega \cap Q_{2^{-k}}) \setminus \partial_\Gamma \Omega \\ w_2 &= -a_k x_n \quad \text{on } \partial_\Gamma \Omega. \end{cases}$$

Now,

- On $\partial_p(\Omega \cap Q_{2^{-k}}) \setminus \partial_\Gamma \Omega$, $v \geq -b_k = w_1 + w_2$,
- On $\partial_\Gamma \Omega \cap \{x_n > 0\}$, $v \geq -a_k x_n = w_2 > w_1 + w_2$,
- On $\{x_n = 0\} \cap \Omega$, $v \geq 0 = w_1 > w_1 + w_2$.

Hence, by the comparison principle, $v \geq w_1 + w_2$ in $\Omega^+_{2^{-k}}$.

Then, we estimate w_1 by the boundary Lipschitz regularity of solutions (for flat boundaries), scaling and linearity to obtain

$$w_1 \geq -Cb_k(2^k x_n) \quad \text{in } Q_{2^{-k-1}} \cap \{x_n > 0\}$$

and we estimate w_2 with the maximum principle with

$$w_2 \geq -a_k \sup_{x \in \partial_\Gamma \Omega \cap Q_{2^{-k}}} \{x_n\} \geq -2^{-k}\omega(2^{-k})a_k.$$

Putting everything together one obtains

$$u \geq a_k x_n + w_1 + w_2 \geq (a_k - 2^k C b_k)x_n - 2^{-k}\omega(2^{-k})a_k \quad \text{in } \Omega^+_{2^{-k-1}}.$$

Moreover, notice that $a_{k+2} = a_{k+1} - C\omega(2^{-k})a_k$,

$$a_2 = a_1 - 2Cb_1 = c_1(1 - 2C\omega(1)) \geq c_1(1 - 2Cc_0) \geq \frac{c_1}{2},$$

and

$$\sum \omega(2^{-k}) \leq \sum \frac{1}{\ln(2)} \int_{2^{-k}}^{2^{-k+1}} \omega(r)\frac{\mathrm{d}r}{r} \leq \frac{c_0}{\ln(2)}.$$

Hence, choosing c_0 small enough we can apply Lemma 5.9.4 and obtain $a_k \geq c_1/12$ for all k. On the other hand, for $k \geq 2$,

$$b_k = 2^{-k+1}a_{k-1}\omega(2^{-k+1}) \leq 2^{-k+1}c_1c_0,$$

using that a_k is decreasing and ω is nondecreasing. Now if we choose $c_0 \leq 1/96$, we have $b_k \leq 2^{-k-2}a_k$ for all $k \geq 2$, and then for all $r \in [2^{-k-1}, 2^{-k})$,

$$u(re_n, 0) \geq a_k r - b_k \geq a_k(r - 2^{-k-2}) \geq \frac{a_k r}{2} \geq \frac{c_1}{24}r,$$

and the conclusion follows. □

Finally, after combining it with the boundary Harnack and the near-linear solution from Section 5.4, we can prove our Hopf lemma for equations with right-hand side.

Proof of Corollary 5.1.11. Let φ be the special solution defined in Proposition 5.4.1, and assume that $\varphi(e_n/2, -1/2) = 1$ after normalizing. From Theorem 5.9.3, for all $r \in (0, \delta)$ and some $c > 0$,

$$\varphi(re_n, 0) \geq cr.$$

Then, divide u by a constant so that $\|u\|_{L^\infty(Q_1)} = 1$. Now we can apply Corollary 5.1.5 to φ and u to obtain

$$\frac{\varphi}{u} \leq C \quad \text{in } Q_{1/2},$$

and hence

$$u(re_n, 0) \geq C^{-1}\varphi(re_n, 0) \geq C^{-1}cr,$$

for all $r \in (0, \min\{1/2, \delta\})$. □

5.10 Appendix: Auxiliary results

5.10.1 The space of the right-hand sides

We will prove an interpolation inequality between weighted L^p spaces that seems classical but we were not able to find in the literature.

Lemma 5.10.1. *Let $\alpha \in (0, 1)$, $p \geq 1$, and let $q = (p + 1)/(1 - \alpha)$. Let $f \in L^q((0, 1))$. Then,*

$$\inf_{\lambda>0} \left[\lambda + \left(\int_0^1 (|f| - \lambda x^{\alpha-1})_+^p x^{-1-p\alpha} \mathrm{d}x\right)^{\frac{1}{p}}\right] \leq 2\|f\|_{L^q((0,1))}.$$

In particular,

$$L^q \subset L^p((0, 1); x^{-\frac{1}{p}-\alpha}) + L^\infty((0, 1); x^{1-\alpha}).$$

Proof. Assume without loss of generality that $f \geq 0$. Then, let us do the change of variables $t = (px^p)^{-1}$, and let us also define $h : (\frac{1}{p}, \infty) \to \mathbb{R}$ as the function satisfying

$$f(x) = x^{\alpha-1}h(t).$$

On the one hand,

$$\|h\|_{L^q}^q = \int_{\frac{1}{p}}^{\infty} h(t)^q \mathrm{d}t = \int_0^1 (f(x)x^{1-\alpha})^q \frac{\mathrm{d}x}{x^{p+1}} = \int_0^1 f(x)^q \mathrm{d}x = \|f\|_{L^q}^q.$$

On the other hand,

$$\begin{aligned}\int_0^1 (f(x) - \lambda x^{\alpha-1})_+^p x^{-1-p\alpha} \mathrm{d}x &= \int_{\frac{1}{p}}^{\infty} (h(t) - \lambda)_+^p x^{p(\alpha-1)} x^{-1-p\alpha} x^{p+1} \mathrm{d}t \\ &= \int_{\frac{1}{p}}^{\infty} (h(t) - \lambda)_+^p \mathrm{d}t = \|(h-\lambda)_+\|_{L^p}^p.\end{aligned}$$

Therefore, it suffices to prove that, given $1 \le p \le q$,

$$\inf_{\lambda>0} [\lambda + \|(h-\lambda)_+\|_{L^p}] \le 2\|h\|_{L^q}.$$

For that purpose, we estimate

$$\|(h-\lambda)_+\|_{L^p}^p \le \int (h-\lambda)_+^p \le \int \chi_{\{h>\lambda\}} \frac{h^q}{\lambda^{q-p}} \le \lambda^{p-q} \int h^q = \lambda^{p-q} \|h\|_{L^q}^q.$$

Then, if we consider $\lambda = \|h\|_{L^q}$,

$$\inf_{\lambda>0} [\lambda + \|(h-\lambda)_+\|_{L^p}] \le \inf_{\lambda>0} \left[\lambda + \lambda^{1-q/p} \|h\|_{L^q}^{q/p}\right] \le 2\|h\|_{L^q}.$$

□

As a consequence, we can interpolate between the weighted Lebesgue spaces used in the one-sided boundary Harnack.

Proposition 5.10.2. *Let $\alpha \in (0,1)$, $q = (n+2)/(1-\alpha)$, and let Ω be a parabolic Lipschitz domain in Q_1 in the sense of Definition 5.2.2. Let $f \in L^q(\Omega)$.*

Then, there exist $g, h : \Omega \to \mathbb{R}$ such that $f = g + h$ and

$$\|d^{1-\alpha} g\|_{L^\infty(\Omega)} + \|d^{-1/(n+1)-\alpha} h\|_{L^{n+1}(\Omega)} \le 2\|f\|_{L^q(\Omega)},$$

where $d(x', x_n, t) = x_n - \Gamma(x', t)$.

Proof. Consider the bi-Lipschitz change of variables $(y', y_n, s) = (x', x_n - \Gamma(x', t), t)$. Then, it suffices to apply Lemma 5.10.1 with $p = n+1$ in the variable y_n and integrate in y' and s. □

5.10.2 The regularized distance

We will give some ideas on how the construction of the regularized distance is done.

Sketch of the proof of Lemma 5.3.2. We follow the construction in [149, Section IV.5]. Let $\varphi \in C^\infty(B_1)$ and $\eta \in C^\infty((0,1))$ be nonnegative cutoff functions with

$$\int_{\mathbb{R}^n} \varphi = \int_{\mathbb{R}} \eta = 1.$$

Let $A = 4\sqrt{L^2+1}$ and $K = \|\eta'\|_{L^1(0,1)}$. We then define

$$F(x,t,\rho) = x_n - \int_{\mathbb{R}^n}\int_{\mathbb{R}} \Gamma\left(x' - \frac{\rho}{A}y, t - \frac{\rho^2}{2(1+K)^2A^2}s\right)\eta(s)\varphi(y)\mathrm{d}s\mathrm{d}y.$$

Recall that $\partial_\Gamma\Omega = \{x_n = \Gamma(x',t)\}$.

Then, it can be shown that for each $(x,t) \in \Omega\cap Q_1$, there is a unique ρ such that $F(x,t,\rho) = \rho$, and we choose $d(x,t)$ to be equal to this ρ.

From the proof in [149, Section IV.5], we obtain

$$\frac{1}{2}(x_n - \Gamma(x',t)) \le d \le \frac{3}{2}(x_n - \Gamma(x',t)),$$
$$\partial_n d \ge \frac{2}{3} \quad \text{and} \quad |\nabla_x d| \le \frac{A}{2}.$$

To obtain the last estimate, one needs to proceed as in [148, Theorem 3.1], notice that since $L \le 1$, $A \in [4, 4\sqrt{2}]$ and it can be absorbed into C_2, and since Γ is not parabolic C^1 but only parabolic Lipschitz, one needs to repeat the computations done with the modulus of continuity of $\nabla_x\Gamma$ substituting it by appropriate expressions concerning the regularity of Γ.

Indeed,

$$|\nabla_x\Gamma(x_1,t_1) - \nabla_x\Gamma(x_2,t_2)| \le \xi(|x_1-x_2|) + \xi(|t_1-t_2|^{1/2})$$

becomes

$$|\nabla_x\Gamma(x_1,t_1) - \nabla_x\Gamma(x_2,t_2)| \le 2\|\nabla_x g\|_{L^\infty} = 2L,$$

and

$$|\Gamma(x,t_1) - \Gamma(x,t_2)| \le |t_1-t_2|^{1/2}\xi(|t_1-t_2|^{1/2})$$

becomes

$$|\Gamma(x,t_1) - \Gamma(x,t_2)| \le L|t_1-t_2|^{1/2}.$$

After these changes, carrying out the rest of the computations in the proof of [148, Theorem 3.1] one can deduce that

$$|\partial_t d| + |D_x^2 d| \le \frac{C_2 L}{d}.$$

□

5.10.3 Blow-up construction

Let us prove how we can construct the blow-up.

Proof of Lemma 5.5.2. First, by the definition of θ, $\theta(r) < \infty$ for each $r > 0$ because $\|u_j\|_{L^\infty(Q_1)} \le 1$, $\|\varphi_j\|_{L^\infty(Q_1)} = 1$, the $K_{r,j}$ are bounded for a fixed r, and by hypothesis $\lim_{r\to 0}\theta(r) = \infty$.

Then, for every positive integer m, there exist $\rho_m \ge 1/m$ and j_m such that

$$\rho_m^{-\beta}\|u_{j_m} - K_{\rho_m,j_m}\varphi_{j_m}\|_{L^\infty(Q_{\rho_m})} \ge \frac{1}{2}\theta(1/m) \ge \frac{1}{2}\theta(\rho_m).$$

Let us choose $\rho_m \downarrow 0$ as follows: if $\theta(1/(m+1)) = \theta(1/m)$, we take $\rho_{m+1} = \rho_m$, and if $\theta(1/(m+1)) > \theta(1/m)$, then there is a suitable $\rho_{m+1} \in [1/(m+1), 1/m)$.

To compute the growth of the w_m, we first need to estimate $\|(K_{2r,j}-K_{r,j})\varphi_j\|_{L^\infty(Q_r)}$. Indeed, using the definition of θ,

$$\begin{aligned}\frac{\|K_{2r,j}\varphi_j - K_{r,j}\varphi_j\|_{L^\infty(Q_r)}}{r^\beta\theta(r)} &\leq \frac{2^\beta\theta(2r)}{\theta(r)}\frac{\|K_{2r,j}\varphi_j - u_j\|_{L^\infty(Q_{2r})}}{(2r)^\beta\theta(2r)} + \frac{\|u_j - K_{r,j}\varphi_j\|_{L^\infty(Q_r)}}{r^\beta\theta(r)} \\ &\leq 2^\beta + 1.\end{aligned}$$

Hence, $\|(K_{2r,j} - K_{r,j})\varphi_j\|_{L^\infty(Q_r)} \leq Cr^\beta\theta(r)$. Analogously, for any $\mu \in [1,2]$,

$$\|(K_{\mu r,j} - K_{r,j})\varphi_j\|_{L^\infty(Q_r)} \leq Cr^\beta\theta(r).$$

Furthermore, given $1 \leq a \leq b$,

$$\begin{aligned}\|(K_{2ar,j} - K_{ar,j})\varphi_j\|_{L^\infty(Q_{br})} &\leq |K_{2ar,j} - K_{ar,j}|\|\varphi_j\|_{L^\infty(Q_{br})} \\ &\leq c_1^{-1}|K_{2ar,j} - K_{ar,j}|(b/a)^\gamma\|\varphi_j\|_{L^\infty(Q_{ar})} \\ &\leq c_1^{-1}(b/a)^\gamma\|(K_{2ar,j} - K_{ar,j})\varphi_j\|_{L^\infty(Q_{ar})} \\ &\leq C(ar)^\beta(b/a)^\gamma\theta(ar) \leq Cr^\beta a^{\beta-\gamma}b^\gamma\theta(r).\end{aligned}$$

Our next step is the following computation. If $R = 2^N R_0 \leq 1/r$ with $R_0 \in [1,2)$,

$$\begin{aligned}\|(K_{Rr,j} - K_{r,j})\varphi_j\|_{L^\infty(Q_{Rr})} &\leq \|(K_{R_0 r,j} - K_{r,j})\varphi_j\|_{L^\infty(Q_{Rr})} \\ &\quad + \sum_{m=0}^{N-1}\|(K_{2^{m+1}R_0 r,j} - K_{2^m R_0 r,j})\varphi_j\|_{L^\infty(Q_{Rr})} \\ &\leq CR^\gamma r^\beta\theta(r) + CR^\gamma r^\beta\theta(r)\sum_{m=0}^{N-1}(2^m R_0)^{\beta-\gamma} \\ &\leq CR^\gamma r^\beta\theta(r) + CR^\gamma r^\beta\theta(r)R^{\beta-\gamma} \leq C(Rr)^\beta\theta(r).\end{aligned}$$

Finally, for any $1 \leq R \leq 1/\rho_m$,

$$\begin{aligned}\|w_m\|_{L^\infty(Q_R)} &= \frac{\|u_{j_m} - K_{\rho_m,j_m}\varphi_{j_m}\|_{L^\infty(Q_{R\rho_m})}}{\|u_{j_m} - K_{\rho_m,j_m}\varphi_{j_m}\|_{L^\infty(Q_{\rho_m})}} \\ &\leq \frac{2\|u_{j_m} - K_{\rho_m,j_m}\varphi_{j_m}\|_{L^\infty(Q_{R\rho_m})}}{\rho_m^\beta\theta(\rho_m)} \\ &\leq \frac{2R^\beta\|u_{j_m} - K_{R\rho_m,j_m}\varphi_{j_m}\|_{L^\infty(Q_{R\rho_m})}}{(R\rho_m)^\beta\theta(\rho_m)} \\ &\quad + \frac{2\|K_{R\rho_m,j_m}\varphi_{j_m} - K_{\rho_m,j_m}\varphi_{j_m}\|_{L^\infty(Q_{R\rho_m})}}{\rho_m^\beta\theta(\rho_m)} \\ &\leq \frac{2R^\beta\theta(R\rho_m)}{\theta(\rho_m)} + CR^\beta \leq CR^\beta.\end{aligned}$$

□

5.10.4 Homogeneous solutions in the complement of *thin cones*

We will prove Proposition 5.6.3. Let us change a bit the notation for convenience of the proof. If φ is a positive solution to

$$\partial_t\varphi - \Delta\varphi = 0 \text{ in } Q_1 \setminus C_\eta,$$

where

$$C_\eta := \{x_n \le \eta(|x'| + |t|^{1/2}),\ x_{n+1} = 0\},$$

and φ satisfies $\varphi(\lambda x, \lambda^2 t) = \lambda^\kappa \varphi(x,t)$, for some κ, then κ is uniquely determined (as a function of η).

Indeed, we can write $\varphi(x,t) = |t|^{\kappa/2}\phi(x/|t|^{1/2})$, and ϕ solves the following eigenvalue problem for the Ornstein-Uhlenbeck operator (see [103, Lemma 5.8]):

$$\begin{cases} \mathcal{L}_{OU}\phi + \frac{\kappa}{2}\phi = 0 & \text{in } \mathbb{R}^{n+1} \setminus \tilde{C}_\eta \\ \phi = 0 & \text{on } \tilde{C}_\eta, \end{cases} \tag{5.8}$$

where

$$\tilde{C}_\eta = \{x_n \le \eta(|x'| + 1),\ x_{n+1} = 0\}$$

and

$$\mathcal{L}_{OU}\phi(x) := \Delta\phi(x) - \frac{x}{2}\cdot\nabla\phi(x) = e^{|x|^2/4}\,\mathrm{Div}(e^{-|x|^2/4}\nabla\phi).$$

Since ϕ is positive, it is the first eigenfunction for $\mathcal{L}_{OU}$ in this domain, and therefore by the Rayleigh quotient characterization,

$$\frac{\kappa}{2} = \inf_{u\in C_c^{0,1}(\mathbb{R}^{n+1}\setminus\tilde{C}_\eta),\ \|u\|_{L^2_w}=1} \int |\nabla u|^2 e^{-|x|^2/4}, \tag{5.9}$$

where

$$\|u\|^2_{L^2_w} := \int u^2 e^{-|x|^2/4}$$

and the infimum is attained by a unique function $\phi_\eta \in L^2_w$ by standard arguments.

Now we are ready to start the proof of Proposition 5.6.3. First we will show the stability of minimizers of the Rayleigh quotient, in the following sense:

Lemma 5.10.3. *Let $\eta \in (-\frac{1}{3}, \frac{1}{3})$ and let $\kappa = \kappa(\eta)$ as in (5.9). Then, for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|u\|_{L^2_w} = 1$, u vanishes on $\tilde{C}_\eta$ and*

$$\int |\nabla u|^2 e^{-|x|^2/4} < \frac{\kappa}{2} + \delta,$$

then

$$\|u - \phi_\eta\|_{L^2_w} < \varepsilon.$$

Proof. Consider the spectral decomposition of $-\mathcal{L}_{OU}$ in the domain $\mathbb{R}^{n+1} \setminus \tilde{C}_\eta$, so that the eigenvalues are

$$0 < \frac{\kappa}{2} < \kappa_2 \le \dots$$

and the eigenfunctions are

$$\{\phi_\eta, \phi_{\eta,2}, \dots\}$$

and form an orthonormal basis with respect to the weighted scalar product

$$\langle f, g\rangle = \int fge^{-|x|^2/4}.$$

Then, if we write $u = \alpha\phi_\eta + \sum c_i\phi_{\eta,i}$,

$$\int |\nabla u|^2 e^{-|x|^2/4} = -\int u\mathcal{L}_{OU}u = \alpha^2\frac{\kappa}{2} + \sum c_i^2\kappa_i < \frac{\kappa}{2} + \delta.$$

On the other hand,

$$1 = \|u\|^2_{L^2_w} = \alpha^2 + \sum c_i^2,$$

and then

$$\sum c_i^2\kappa_i < \sum c_i^2\frac{\kappa}{2} + \delta \;\Rightarrow\; \sum c_i^2\left(\kappa_i - \frac{\kappa}{2}\right) < \delta,$$

so it follows that

$$\|u - \phi_\eta\|^2_{L^2_w} = 2\sum c_i^2 < \frac{\delta}{\kappa_2 - \kappa/2} < \varepsilon,$$

as required. □

Then, we see the monotonicity and continuity of the eigenvalue with respect to the domain.

Lemma 5.10.4. *Let $\kappa : (-\frac{1}{3}, \frac{1}{3}) \to \mathbb{R}$ as in (5.9). Then, κ is strictly increasing and continuous.*

Proof. First we prove the monotonicity. Let $-\frac{1}{3} < \eta_1 < \eta_2 < \frac{1}{3}$. Then, $\tilde{C}_{\eta_1} \subset \tilde{C}_{\eta_2}$, so $\mathbb{R}^{n+1} \setminus \tilde{C}_{\eta_2} \subset \mathbb{R}^{n+1} \setminus \tilde{C}_{\eta_1}$, and since the infimum in the Rayleigh quotient is taken over more functions in the case of η_1, we get $\kappa(\eta_1) \le \kappa(\eta_2)$.

Now, if $\kappa(\eta_1) = \kappa(\eta_2)$, this means that ϕ_{η_2} is a solution to (5.8) with η_1. But ϕ_{η_2} is identically zero in $\tilde{C}_{\eta_2} \setminus \tilde{C}_{\eta_1}$, and hence we have a solution to an elliptic equation that vanishes in an open subset of the domain, contradicting the strong maximum principle. Therefore it cannot be $\kappa(\eta_1) = \kappa(\eta_2)$ and it must be $\kappa(\eta_1) < \kappa(\eta_2)$.

On the other hand, to prove continuity we will obtain an upper bound for $\kappa(\eta_2)$ in terms of $\kappa(\eta_1)$ by deforming the domain and the solution for η_1 to get a competitor.

Let

$$\alpha = \arctan(\eta_2) - \arctan(\eta_1) < 2\arctan(1/3) < \pi/3,$$

and define $\tau : \mathbb{S}^1 \to \mathbb{S}^1$ as

$$\tau(\theta) = \begin{cases} \left(1 + \frac{3\alpha}{\pi}\right)\theta & |\theta| \le \frac{\pi}{3}, \\ \theta + \alpha\,\mathrm{sgn}(\theta) & \frac{\pi}{3} < |\theta| \le \frac{2\pi}{3}, \\ \left(1 - \frac{3\alpha}{\pi}\right)\theta + 3\alpha\,\mathrm{sgn}(\theta) & \frac{2\pi}{3} < |\theta|. \end{cases}$$

Then, let $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ be defined in polar coordinates as $\rho(r,\theta) = (r, \tau(\theta))$, and let $J : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $J(x,y) = (y,x)$. Thus,

$$\psi(x_1, \ldots, x_{n-1}, x_n, x_{n+1}) := \phi_{\eta_1}(x_1, \ldots, J(\rho(J(x_{n-1}, x_n))), x_{n+1})$$

is a positive function that vanishes on $\tilde{C}_{\eta_2}$, and we can get an upper bound for $\kappa(\eta_2)$ by computing the value of its Rayleigh quotient, i.e.

$$\frac{\kappa(\eta_2)}{2} \le \frac{1}{\|\psi\|_{L^2_w}} \int |\nabla\psi|^2 e^{-|x|^2/4}.$$

Observe that the deformation of space introduced by ρ changes volume in a factor of $1+\frac{3\alpha}{\pi}$, 1 or $1-\frac{3\alpha}{\pi}$. Hence, by scaling we obtain the following estimates:

$$\|\psi\|_{L^2_w} \geq \left(1-\frac{3\alpha}{\pi}\right)\|\phi_{\eta_1}\|_{L^2_w} = 1-\frac{3\alpha}{\pi}$$

and

$$\int |\nabla\psi|^2 e^{-|x|^2/4} \leq \left(1-\frac{3\alpha}{\pi}\right)^{-1}\int |\nabla\phi_{\eta_1}|^2 e^{-|x|^2/4} \leq \left(1-\frac{3\alpha}{\pi}\right)^{-1}\frac{\kappa(\eta_1)}{2},$$

and combining them we get

$$\kappa(\eta_1) < \kappa(\eta_2) \leq \left(1-\frac{3}{\pi}(\arctan(\eta_2)-\arctan(\eta_1))\right)^{-2}\kappa(\eta_1),$$

which already implies that κ is continuous. $\square$

To conclude, we write the following:

Proof of Proposition 5.6.3. Let $\kappa : (-\frac{1}{3},\frac{1}{3}) \to \mathbb{R}$ as in Lemma 5.10.4, that is strictly increasing and continuous, and observe that $\kappa(0)=1$ because $\varphi_0(x,t) := \mathrm{Re}(\sqrt{x_n + ix_{n+1}})$ is the solution to the original parabolic problem for $\eta = 0$.

Then, there exists $\varepsilon_0 > 0$ such that $\kappa^{-1} : (1-2\varepsilon_0, 1+2\varepsilon_0) \to [-\frac{1}{4},\frac{1}{4}]$ is well defined, continuous and strictly increasing. Moreover, for any $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$, let

$$\varphi_\varepsilon(x,t) := \frac{\tilde\varphi_\varepsilon}{\|\tilde\varphi_\varepsilon\|_{L^\infty(Q_1)}} := \frac{|t|^{1/2+\varepsilon}\phi_\eta(x/|t|^{1/2})}{\||t|^{1/2+\varepsilon}\phi_\eta(x/|t|^{1/2})\|_{L^\infty(Q_1)}},$$

where $\eta = \kappa^{-1}(1+2\varepsilon)$. To see that φ_ε is well defined, we need to check that $\tilde\varphi_\varepsilon \in L^\infty(Q_1)$ and that it is not identically zero in Q_1.

First, recall that $\tilde\varphi_\varepsilon$ is a positive solution to the heat equation in $\mathbb{R}^{n+2}\setminus C_\eta$ that vanishes on C_η, so in particular it is a subsolution in the full space. Now, $\tilde\varphi_\varepsilon(\cdot,-1) \equiv \phi_\eta$, and then, for all $(x,t) \in Q_{1/2}$,

$$\begin{aligned}\tilde\varphi_\varepsilon(x,t) &\leq C\int \phi_\eta(y)e^{-|x-y|^2/4} \leq C\left(\int e^{-|x-y|^2/6}\right)^{1/2}\left(\int \phi_\eta(y)^2 e^{-|x-y|^2/3}\right)^{1/2}\\ &\leq C\left(\int \phi_\eta(y)^2 e^{-|y|^2/4}\right)^{1/2} = C,\end{aligned}$$

where we used that for all $x \in B_{1/2}$,

$$-\frac{|x-y|^2}{3} \leq C - \frac{|y|^2}{4}.$$

Hence, by homogeneity, $\|\tilde\varphi_\varepsilon\|_{L^\infty(Q_1)} \leq 2^{1/2+\varepsilon}C$.

On the other hand, since for all $-\frac{1}{3} < \eta_1 < \eta_2 < \frac{1}{3}$, ϕ_{η_2} vanishes on $\tilde C_{\eta_1}$, by Lemmas 5.10.3 and 5.10.4,

$$\lim_{\eta_2\to\eta_1}\|\phi_{\eta_2}-\phi_{\eta_1}\|_{L^2_w} = 0.$$

Now, let $E = B_1 \cap \{|x_{n+1}| \geq \frac{1}{4(n+1)}\}$. We claim that, for all $\eta \in [-\frac{1}{4},\frac{1}{4}]$, $\|\phi_\eta\|_{L^\infty(E)} \geq c > 0$, where c is independent of η. Let us prove it by contradiction. If not, there would exist $\{\eta_k\}$

such that $\|\phi_{\eta_k}\|_{L^\infty(E)} < 1/k$, and therefore, after choosing a subsequence, $\eta_{k_m} \to \eta_0 \in [-\frac{1}{4}, \frac{1}{4}]$, and by continuity $\phi_{\eta_{k_m}} \to \phi_{\eta_0}$ strongly in L^2_w, and hence $\phi_{\eta_0} \equiv 0$ in E, contradicting the strong maximum principle.

Furthermore, by the parabolic interior Harnack inequality, $\tilde{\varphi}_\varepsilon \geq c'$ in the set $Q_{1/2} \cap \{x_{n+1} \geq \frac{1}{4(n+1)}\}$, and by homogeneity $\tilde{\varphi}_\varepsilon \geq c'$ also in $Q_2 \cap \{|x_{n+1}| \geq \frac{1}{n+1}\}$.

Therefore, for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $\tilde{\varphi}_\varepsilon \geq c'$ in $Q_2 \cap \{|x_{n+1}| \geq \frac{1}{n+1}\}$ and $\|\tilde{\varphi}_\varepsilon\|_{L^\infty(Q_1)} \leq C$, and thus the conclusion follows. $\square$

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